Closure Operations in Positive Characteristic and Big Cohen-Macaulay Algebras

by

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I finally give thanks to God for blessing me with the opportunity to learn about and study the beauty inherent in mathematics.

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ABSTRACT

Closure Operations in Positive Characteristic and Big Cohen-Macaulay Algebras

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In this thesis, we investigate some open questions involving tight closure and big Cohen-Macaulay algebras over rings of positive prime characteristic. We show that if R is a standard graded domain over an algebraically closed field where tight closure equals graded-plus closure for finitely generated graded modules, then the equivalence also holds for non-graded finitely generated modules with a quotient coprimary to the homogeneous maximal ideal of R. We also provide a construction of an R-algebra B such that B is a degree-preserving big Cohen-Macaulay algebra over R and so that an element x of R is in the tight closure of an ideal I only if x is in the expansion of I to B.

We additionally examine the class of algebras over a local ring R that map to some big Cohen-Macaulay R-algebra; we call such rings "seeds" over R. Our results include that tensor products of seeds are seeds and that the seed property is preserved under base change between complete local domains of positive characteristic. These results are used to answer some open questions involving big Cohen-Macaulay algebras over complete local domains of positive characteristic. We also provide a family of equational problems that classify the obstruction for all solid algebras to be seeds over complete local domains in positive characteristic, and we characterize solid algebras in terms of phantom extensions.

Finally, we discuss a program to axiomatically define a closure operation that will be a good analogue of tight closure.

CHAPTER 1

Introduction

Since the 1970s, when C. Peskine and L. Szpiro published proofs of the local homological conjectures in prime characteristic $p > 0$ (see [PS1] and [PS2]), positive characteristic methods have been a popular, and, more importantly, successful tool for solving problems in commutative algebra and algebraic geometry, prompting several modern commutative algebraists to aver, "Life in characteristic p is better!" The primary ingredient in a characteristic p method is the use of the *Frobenius endomorphism* $F : R \to R$ that takes $r \mapsto r^p$ in rings containing a field of positive characteristic. The existence and exploitation of this map has had profound impact in the world of commutative algebra.

In particular, *tight closure* theory, since its introduction by M. Hochster and C. Huneke in the late 1980s, has been an important method of working in positive characteristic. The tight closure of an ideal is defined to be

$$I^* := \{x \in R \,|\, \exists c \in R^\circ \text{ such that } cx^{p^e} \in I^{[p^e]} \quad \forall e \gg 0\}$$

where p is the characteristic of the ring, R° is the complement in R of the minimal primes of R, and $I^{[p^e]}$ is the ideal generated by all p^eth powers of the elements of I. One might view tight tight closure theory as a unifying theory for many characteristic p methods, as the success of tight closure has been due to its ability to tie together ideas that were previously not known to be connected, generalize theorems, and simplify proofs. Some of the many examples include proofs that the integral closure of the n^{th} power of an ideal I in a regular ring is contained in I (the Briançon-Skoda Theorem) and that direct summands of regular rings are Cohen-Macaulay (a generalization of the Hochster-Roberts Theorem).

Methods discovered by Hochster and Huneke during their development of tight closure also led to their remarkable result that the absolute integral closure R^+ of an excellent local domain R of positive characteristic is a *big Cohen-Macaulay algebra* over R; see [HH2]. A big Cohen-Macaulay algebra B is an R-algebra for which every system of parameters of R is a regular sequence on B. While Hochster had previously shown the existence of big Cohen-Macaulay modules in equal characteristic (see [Ho2]), this new result was the first proof that big Cohen-Macaulay algebras existed. Their existence is important as it gives new proofs for many of the homological conjectures, such as the direct summand conjecture, monomial conjecture, and vanishing conjecture for maps of Tor in positive characteristic. As with tight closure, the use of the existence of big Cohen-Macaulay algebras is remarkable because it can help prove theorems that make no explicit reference to big Cohen-Macaulay algebras.

In this thesis, we will study some open questions involving tight closure, including whether $I^* = IR^+ \cap R$, for certain rings. We will also present new properties about big Cohen-Macaulay algebras in positive characteristic and the rings that map to such algebras. Closely related to the topics of tight closure and big Cohen-Macaulay algebras is the class of *solid algebras* defined by Hochster in [Ho3]. We will also investigate some of the connections between these areas and try to shed more light on their relationship.

In Chapter 2, we will review the necessary background material used to motivate and provide the fundamental tools used for proving the results of the succeeding chapters.

Tight closure of finite length modules in graded rings

At the forefront of unresolved problems involving tight closure stands the localization question. Deciding whether tight closure computations commute with localization or not remains a very elusive goal. Closely related to the localization issue is the question of whether the tight closure of an ideal I in a positive characteristic domain R is simply the contracted-expansion of I to R^+, the integral closure of R in an algebraic closure of its fraction field, i.e., does $I^* = IR^+ \cap R$? (The ideal $IR^+ \cap R$ is called the *plus closure* of I.) Since the plus closure operation can easily be shown to commute with localization, one could settle the localization question, while also giving a simple alternative definition of tight closure, by proving the above equality.

In the early 1990s, K.E. Smith made a tremendous contribution to this problem by proving that $I^* = IR^+ \cap R$ for ideals generated by partial systems of parameters in excellent local domains of positive characteristic; see [Sm1]. Smith also showed that tight closure and plus closure are equal for ideals generated by part of a homogeneous system of parameters in an $\mathbb{N}$-graded domain R finitely generated over R_0 when R_0 is a field of positive characteristic; see [Sm2].

Very recently, H. Brenner made a major breakthrough on this problem when he showed that tight closure and plus closure are equivalent for homogeneous ideals in certain 2-dimensional graded rings; see [Br2] and [Br3]. Specifically, the equivalence holds for graded ideals when the ring is either an $\mathbb{N}$-graded 2-dimensional domain of finite type over the algebraic closure of a finite field or the homogeneous coordinate ring of an elliptic curve over an algebraically closed field. In both cases, Brenner's works relies heavily on a correspondence between tight closure and projective bundles (see [Br1]), which allows one to use machinery from algebraic geometry to develop numerical criteria that determine when an element is in the tight closure or plus closure of an ideal. Although it is not made explicit in his work, it appears that the same methods show the equivalence for finitely generated graded modules over the same class of rings. See Section 3.2 for details.

Inspired by Brenner's recent progress, we have studied how one can obtain an equivalence of tight closure and plus closure for more general ideals and modules given that one has the equivalence for homogeneous ideals and modules. While we have not yet been able to obtain an extension to all modules, in Chapter 3 we prove:

Theorem 3.1.6. *Let (R, m) be a standard graded K-algebra of characteristic $p > 0$. Suppose that R is a domain, and K is algebraically closed. If $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$ for all finitely generated graded R-modules $N \subseteq M$ such that M/N is m-coprimary, then the same is true for* all *finitely generated modules $N \subseteq M$ such that M/N is m-coprimary.*

As a result, we can apply our theorem to the cases where Brenner's work is valid to increase the class of ideals and modules where tight closure equals plus closure.

Unlike the work of Brenner, our methods are entirely algebraic and rely on injective modules over a graded subring, $R^{+\mathrm{GR}}$, of R^+. This led us to the study of injective hulls over $R^{+\mathrm{GR}}$ and R^∞ in an attempt to extend our result beyond the m-coprimary case. We manage to compute a submodule of the injective hull $E_{A^\infty}(K)$, where A is either a polynomial ring or a formal power series ring, that enables us to show that the injective hulls $E_{R^{+\mathrm{GR}}}(K)$ and $E_{R^+}(K)$ behave far differently from the Noetherian case, as these modules contain elements that are not killed by any power of the maximal ideal of R; see Proposition 3.3.13. As a result we cannot use these injective hulls to extend Theorem 3.1.6 to general modules.

Graded-complete rings and modules

In Chapter 4, we investigate another problem involving tight closure in graded rings. In [Ho3, Theorem 11.1], Hochster provides a construction of big Cohen-Macaulay algebras which implicitly shows that one can construct a big Cohen-Macaulay algebra B over a complete local domain R of positive characteristic such that $I^* = IB \cap R$, for all ideals, with a similar result for finitely generated modules. We have used a similar process to show that one can construct a big Cohen-Macaulay algebra B over a standard graded domain R that also "captures" tight closure; see Theorem 4.5.9 and Corollary 4.6.1.

Since the construction of B required us to go beyond the realm of graded rings and since we still wanted B to possess a notion of "degree," we found it necessary to produce a class of rings and modules that we have termed *graded-complete*. See Sections 4.1 and 4.2 for definitions. In a sense, these are rings and modules that possess a well-defined notion of degree, like graded objects, but also possess properties of complete rings, such as infinite formal sums. In the case that a ring R is $\mathbb{N}$-graded and finitely generated over $R_0 = K$, a field, the graded-completion operation coincides with completion at the homogeneous maximal ideal.

Solid algebras

Developed by Hochster in [Ho3], *solid modules* and *algebras* have many connections to tight closure in positive characteristic and to big Cohen-Macaulay algebras. Hochster defined an R-module (not necessarily finitely generated) over a domain R to be *solid* if $\mathrm{Hom}_R(M, R) \neq 0$. If $M = S$ is also an R-algebra, then S is called a *solid algebra*. What may appear at first to be a minor non-degeneracy condition on an R-algebra is actually a condition that allows one to give an alternate characterization of tight closure in positive characteristic. Briefly, Hochster shows in [Ho3, Theorem 8.6] that over a complete local domain R of positive characteristic, $u \in I^*$ if and only if $u \in IS \cap R$, for some solid R-algebra S. He uses this notion of contracted-expansion from some solid algebra as the basis for a closure operation dubbed *solid closure*.

Another significant aspect of solid algebras is given in [Ho3, Corollary 10.6], which shows that an algebra over a complete local domain R (in any characteristic) that maps to a big Cohen-Macaulay algebra is solid. The question of whether the converse is true is open in positive and mixed characteristic. Hochster shows that it is true in general when $\dim R \leq 2$ and that it is false in equal characteristic 0.

In Chapter 5 we examine this question and manage to produce a family of finite type algebras $A_p^{(n,k,d)}$ over a finite field of characteristic p, parameterized by four nonnegative integers, for which the converse will be true in positive characteristic if the local cohomology modules $H_m^d(A_p^{(n,k,d)})$ all vanish; see Proposition 5.1.16. This reduction of the problem puts the issue on an entirely computational footing and potentially allows one to develop new answers using computer algebra software, such as *Macaulay 2* (see [M2] for more information). Using this approach, we show that the relevant local cohomology modules vanish when $n \leq 2$, $k = 0$, or $d \leq 1$, which provides new evidence that all solid algebras may map to big Cohen-Macaulay algebras in positive characteristic. See Proposition 5.1.23. Attempts to use *Macaulay 2* for calculations in the algebra $A_2^{(3,1,2)}$, the next open case, have been stymied by the fact that the computations involved have been too complicated for the software to handle.

In another attempt to make new progress on whether solid algebras map to big Cohen-Macaulay algebras, we present new characterizations of solid algebras over complete local domains of positive characteristic using the notion of *phantom extensions* developed by Hochster and Huneke in [HH5]. A map $\alpha : N \to M$ of R-modules is a *phantom extension* if there exists $c \in R^\circ$ such that for all $e \gg 0$, there exists a map $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$. Using this definition, we show in Theorem 5.2.13 that an R-algebra is solid if and only if it is a phantom extension of R if and only if it is a direct limit of module-finite phantom extensions of R. Despite this explicit connection between solid algebras and phantom extensions, we have not yet been able to use this characterization to make new progress on whether solid algebras map to big Cohen-Macaulay algebras, even though [HH5, Section 5] shows that phantom extensions can be used to construct big Cohen-Macaulay modules.

Algebras that map to big Cohen-Macaulay algebras

In what may be the centerpiece of this thesis, we delve into the properties possessed by algebras, which we have termed *seeds*, that map to big Cohen-Macaulay algebras. In Chapter 6 we scrutinize seeds and produce some new properties involving big Cohen-Macaulay algebras. We will show that over a complete local domain of positive characteristic any two big Cohen-Macaulay algebras map to a common big Cohen-Macaulay algebra. We will also strengthen the "weakly functorial" existence result of Hochster and Huneke ([HH7, Theorem 3.9]) by showing that the seed property is stable under base change between complete local domains of positive characteristic.

One of our most useful results is given in Theorem 6.4.8, where we show that if R is a local Noetherian ring of positive characteristic, S is a seed over R, and T is an integral extension of S, then T is also a seed over R. We can even view this theorem as a generalization of the existence of big Cohen-Macaulay algebras over complete local domains of positive characteristic since this existence follows as a corollary; see Corollary 6.4.10. The other results discussed below from Chapter 6 are also proved using Theorem 6.4.8.

We also define a class of *minimal seeds*, which, analogously to the *minimal solid* algebras of Hochster, are seeds that have no proper homomorphic image that is also a seed. While it has only been shown that Noetherian solid algebras map onto minimal solid algebras in [Ho3], we have shown in Proposition 6.3.4 that every seed maps onto a minimal seed. Furthermore, like minimal solid algebras, we have shown that minimal seeds are domains in positive characteristic; see Proposition 6.5.2. As a result, every seed in positive characteristic over a local ring (R, m) maps to a big Cohen-Macaulay algebra that is an absolutely integrally closed, m-adically separated, quasilocal domain; see Proposition 6.5.8.

Perhaps the most interesting results of Chapter 6 are Theorems 6.6.4 and 6.6.10. Both results concern seeds over a complete local domain R of positive characteristic. Theorem 6.6.4 shows that the tensor product of two seeds is still a seed, just as the tensor product of two solid modules is still solid. As an immediate consequence, if B and B' are big Cohen-Macaulay algebras over R, then B and B' both map to a common big Cohen-Macaulay algebra C, which shows that, in a sense, the class of big Cohen-Macaulay algebras over R forms a directed system.

Theorem 6.6.10 shows that if $R \to S$ is a map of positive characteristic complete local domains, and T is a seed over R, then $T \otimes_R S$ is a seed over S, so that the seed property is stable under this manner of base change. This theorem also gives an improvement on the "weakly functorial" result of Hochster and Huneke in [HH7, Theorem 3.9], where they show that given complete local domains of equal characteristic $R \to S$, there exists a big Cohen-Macaulay R-algebra B and big Cohen-

Macaulay S-algebra C such that $B \to C$ extends the map $R \to S$. Our theorem shows that if we have $R \to S$ in positive characteristic, and B is any big Cohen-Macaulay R-algebra, then there exists a big Cohen-Macaulay S-algebra C that fills in a commutative square:

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

Axioms for a good closure operation

Chapter 7 begins an investigation of what axioms a closure operation must possess to imply a "weakly functorial" existence of big Cohen-Macaulay algebras. This is made precise at the beginning of Chapter 7.

Based upon Theorems 6.6.4 and 6.6.10 of Chapter 6, we can use the class of all big Cohen-Macaulay algebras over complete local domains R of positive characteristic to define a closure operation, which we denote by $N^{\natural}_M$ and call $\natural$-*closure*. For an ideal I of R, an element u is in $I^{\natural}$ if and only if $u \in IB$, for some big Cohen-Macaulay R-algebra B, with a similar definition for modules. While [Ho3, Theorem 11.1] implies that this closure operation is equivalent to tight closure over complete local domains of positive characteristic, our results allow us to prove *directly* that $\natural$-closure has many of the nice properties of tight closure.

We present a list of axioms for a closure operation in Section 7.1, which we believe at least form the nucleus for a set of axioms that will be strong enough to prove the weakly functorial existence of big Cohen-Macaulay algebras. We also demonstrate in Section 7.4 that the weakly functorial existence of big Cohen-Macaulay algebras induces a closure operation that satisfies all of our axioms, supporting the idea that these are "good" axioms, although more may be needed.

CHAPTER 2

Background Information

Before we state and prove our results, we will survey some of the theory that forms the foundation and provides motivation for our work.

All rings throughout are commutative with identity and are Noetherian unless noted otherwise. All modules are unital.

2.1 The Frobenius Endomorphism

Of central importance to the work of this thesis are rings of positive prime characteristic p. In order to simplify notation, we will let e denote a nonnegative integer and let q denote p^e, a power of p. Thus, the phrase "for all q" will mean "for all powers $q = p^e$ of p."

Each characteristic p ring R comes equipped with an endomorphism $F = F_R$ mapping $R \to R$, called the *Frobenius endomorphism*, which maps $r \mapsto r^p$. We can compose this map with itself to obtain the iterations $F^e = F_R^e : R \to R$, which map $r \mapsto r^q$.

Closely associated to these maps are the *Peskine-Szpiro* (or *Frobenius*) *functors* $\mathbf{F}^e = \mathbf{F}_R^e$. If we let S denote the ring R viewed as an R-module via the e^{th}-iterated Frobenius endomorphism, then $\mathbf{F}^e$ is the covariant functor $S \otimes_R -$ which takes R-modules to S-modules and so takes R-modules to R-modules since $S = R$ as a ring.

Specifically, if $R^m \to R^n$ is a map of free R-modules given by the matrix (r_{ij}), then we may apply $\mathbf{F}^e$ to this map to obtain a map between the same R-modules given by the matrix (r_{ij}^q). For cyclic modules R/I, $\mathbf{F}^e(R/I) = R/I^{[q]}$, where

$$I^{[q]} := (a^q \mid a \in I)R$$

is the q^{th} *Frobenius power* of the ideal I. If the ideal I is finitely generated, then $I^{[q]}$ is also the ideal generated by the q^{th} powers of a finite generating set for I. In a similar manner, for modules $N \subseteq M$, we will denote the image of $\mathbf{F}^e(N)$ in $\mathbf{F}^e(M)$ by $N_M^{[q]}$, and we will denote the image of $u \in N$ inside of $N_M^{[q]}$ by u^q.

When the ring R is reduced, we can define $R^{1/q}$ to be the ring obtained by adjoining to R all q^{th} roots of elements in R. In this setting, the inclusion $R \hookrightarrow R^{1/q}$ is isomorphic to the inclusion $F^e : R \hookrightarrow R$, identifying $R^{1/q}$ with R via the isomorphism $r^{1/q} \mapsto r$. Therefore the Peskine-Szpiro functor $\mathbf{F}^e$ is isomorphic to $R^{1/q} \otimes_R -$, and the Frobenius power $I^{[q]}$ can be identified with the extension $IR^{1/q}$. We will also make use of the *perfect closure* of R, which we denote by R^∞. The ring R^∞ is constructed by adjoining all q^{th} roots to R, for all q. We can also view R^∞ as the direct limit of the rings $R^{1/q}$ under the natural inclusion maps $R \hookrightarrow R^{1/p} \hookrightarrow R^{1/p^2} \hookrightarrow \cdots \hookrightarrow R^{1/q} \hookrightarrow \cdots$. In general, R^∞ is not a Noetherian ring.

Using the Peskine-Szpiro functors, one can define a closure operation for ideals and modules called the *Frobenius closure*.

Definition 2.1.1. For a Noetherian ring R of positive characteristic p and finitely generated R-modules $N \subseteq M$, the *Frobenius closure* of N in M is the submodule

$$N_M^F := \{u \in M \mid u^q \in N_M^{[q]}, \text{ for some } q\}.$$

In the case that R is reduced, we may also characterize the Frobenius closure as in the following lemma.

Lemma 2.1.2. *If R is a reduced Noetherian ring of characteristic $p > 0$, then for finitely generated R-modules $N \subseteq M$, the following are equivalent:*

(i) $u \in N_M^F$.

(ii) $1 \otimes u \in \operatorname{Im}(R^{1/q} \otimes N \to R^{1/q} \otimes M)$, *for some q.*

(iii) $1 \otimes u \in \operatorname{Im}(R^{\infty} \otimes N \to R^{\infty} \otimes M)$.

In the case of ideals, $I^F = \bigcup_q IR^{1/q} \cap R = IR^{\infty} \cap R$.

It is clear from the definition that N_M^F is a submodule of M. We will see in the upcoming section that the Frobenius closure is smaller in general than the tight closure, but we will see some cases in Chapter 3 where the two notions are equivalent.

2.2 Tight Closure

Perhaps the most prominent tool in the class of characteristic p methods is tight closure. The operation of tight closure was developed by M. Hochster and C. Huneke in the late 1980s and early 1990s as a method for proving (often reproving with dramatically shorter proofs) and generalizing theorems for commutative rings containing a positive characteristic field. Although the definition of tight closure relies on the Frobenius endomorphism, they have also developed a notion of characteristic 0 tight closure using methods of reduction to characteristic p that allows one to carry many results proved in positive characteristic using tight closure over to the case of rings containing a copy of the rational numbers. We shall not deal with the characteristic 0 theory but refer the interested reader to [HH8] for a detailed treatment.

We will now provide a brief overview of the theory of tight closure and highlight many of the results that will be useful in our later studies. Throughout the rest of this section, we will assume that R is a positive characteristic, Noetherian ring. We will denote the complement in R of the set of minimal primes by R°.

Definition 2.2.1. For a Noetherian ring R of characteristic $p > 0$ and finitely generated modules $N \subseteq M$, the *tight closure* N_M^* of N in M is

$$N_M^* := \{u \in M \mid cu^q \in N_M^{[q]} \text{ for all } q \gg 1, \text{ for some } c \in R^\circ\}.$$

If $N_M^* = N$, then N is *tightly closed* in M.

In the case that $M = R$ and $N = I$, $u \in I^*$ if and only if there exists $c \in R^\circ$ such that $cu^q \in I^{[q]}$, for all $q \gg 1$.

Remark 2.2.2. Proposition 8.5 of [HH1] shows that if R is reduced, then we can replace "$q \gg 1$" above with "$q \geq 1$."

It is important to notice that a different c may be chosen for each $u \in I^*$, but that the same c is used for all values of q. We shall discuss a little bit later the notion and existence of *test elements*, which are elements c that can be used in every tight closure computation.

We now record a number of useful facts about tight closure, all of which can be found in [HH1].

Proposition 2.2.3. *Let R be a Noetherian ring of characteristic p, let $N \subseteq M$ and W be finitely generated modules, and let I be an ideal.*

(a) *N_M^* is a submodule of M containing N.*

(b) *If $M \subseteq W$, then $N_W^* \subseteq M_W^*$.*

(c) *If $f : M \to W$, then $f(N_M^*) \subseteq f(N)_W^*$.*

(d) *$(IN)_M^* = (I^*N^*)_M^*$.*

(e) *$(N_M^*)_M^* = N_M^*$.*

(f) *$u \in N_M^*$ if and only if $u + N \in 0_{M/N}^*$.*

(g) *$N_M^F \subseteq N_M^*$, and for ideals $I^F \subseteq I^* \subseteq \overline{I}$, where $\overline{I}$ is integral closure of I.*

(h) *$u \in N^*_M$ if and only if $u + \mathfrak{p}M \in \mathrm{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)^*_{M/\mathfrak{p}M}$, calculated over $R/\mathfrak{p}R$, for all minimal primes $\mathfrak{p}$ of R.*

(i) *If R is regular, then $N^*_M = N$.*

(j) *If R, N, and M are graded such that the inclusion preserves degree, then N^*_M is also graded.*

(k) $(0)^* = \mathrm{Rad}(0)$. *In particular, if R is reduced,* $(0)^* = (0)$.

Any ring R in which every ideal (and thus finitely generated module) is tightly closed is called *weakly F-regular*. If every localization of R is weakly F-regular, then R is called *F-regular*. Therefore, (i) above states that regular rings are F-regular. The "weakly" terminology is due to the fact that we do not know whether tight closure commutes with localization or not. This localization question is the most outstanding open problem in the theory of tight closure, and a lack of an answer often leads to cumbersome notation or hypotheses. For some impressive progress on the localization question; see [AHH].

One of the triumphs of tight closure has been its ability to generalize previously known theorems while also simplifying their proofs. A primary example is the Briançon-Skoda Theorem, which is reproved using tight closure in [HH1, Theorem 5.4]. The original theorem was proved for regular rings in equal characteristic 0 using very lengthy and involved analytic techniques.

Theorem 2.2.4 (generalized Briançon-Skoda Theorem, [HH1])**.** *Let R be a Noetherian ring of positive characteristic p, and let I be an ideal of positive height generated by n elements. Then $\overline{I^{n+m}} \subseteq (I^{m+1})^*$, for any $m \geq 0$.*

A very powerful tool in the application of tight closure is the notion of a *test element*. As mentioned earlier, these are elements that can be used in all tight closure computations. Many proofs obtained using tight closure methods rely on the existence of test elements. Fortunately, we will soon see that test elements exist in very general classes of rings.

Definition 2.2.5. For a Noetherian ring of positive characteristic p, an element $c \in R^\circ$ is called a *test element* if, for all ideals I and all $u \in I^*$, we have $cu^q \in I^{[q]}$ for all $q \geq 1$. If, further, c is a test element in every localization of R, then c is *locally stable*, and if it is also a test element in the completion of every localization, then c is *completely stable*.

As stated above, test elements exist in very general settings. Specifically, Hochster and Huneke proved the following statement.

Theorem 2.2.6 (Theorem 6.1, [HH6])**.** *If R is a reduced, excellent local ring of positive characteristic, then R has a completely stable test element.*

The assumption that R be excellent is extremely mild as excellent rings are ubiquitous in commutative algebra and algebraic geometry. The situation in the graded case is also very good.

Theorem 2.2.7 (Theorem 4.2d, [HH5])**.** *If R is an $\mathbb{N}$-graded, reduced Noetherian ring of positive characteristic, and R_0 is a field, then R possesses a homogeneous completely stable test element.*

Using the existence of test elements, we can produce some easy characterizations of tight closure using $R^{1/q}$ and R^∞.

Lemma 2.2.8. *Let $N \subseteq M$ be finitely generated R-modules, where R is reduced of positive characteristic p. Then the following are equivalent:*

(i) $u \in N^*_M$.

(ii) $c \otimes u \in N_M^{[q]}$, *for all* $q \gg 0$ *and some* $c \in R^\circ$.

(iii) $c^{1/q} \otimes u \in \mathrm{Im}(R^{1/q} \otimes N \to R^{1/q} \otimes M)$, *for all* $q \gg 0$ *and some* $c \in R^\circ$.

(iv) $c^{1/q} \otimes u \in \mathrm{Im}(R^\infty \otimes N \to R^\infty \otimes M)$, *for some test element* c.

(v) $c^{1/q} \otimes u \in \mathrm{Im}(R^\infty \otimes N \to R^\infty \otimes M)$, *for all test elements* c.

As another consequence of the existence of test elements, Hochster and Huneke established the following base change result, which they call *persistence*.

Theorem 2.2.9 (Theorem 6.24, [HH6])**.** *Let* $R \to S$ *be a homomorphism of excellent Noetherian rings of positive characteristic. If* $N \subseteq M$ *are finitely generated* R*-modules, and* $w \in N_M^*$, *then* $1 \otimes w \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^*_{S \otimes_R M}$.

It will be worthwhile to note that somewhat of a converse result can be obtained when the map $R \to S$ is a module-finite extension.

Theorem 2.2.10 (Corollary 5.23, [HH5])**.** *Let* S *be a module-finite extension of a positive characteristic Noetherian ring* R*. If* $1 \otimes u \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^*_{S \otimes_R M}$, *calculated over* S*, then* $u \in N_M^*$*. In particular,* $(IS)^*_S \cap R \subseteq I^*_R$ *for all ideals* I *of* R.

Two other important notions in the theory of tight closure are colon-capturing and phantom acyclicity. One can think of phantom acyclicity as a tight closure analogue of ordinary acyclicity of a complex of modules. Instead of requiring that the cycles be contained in the boundaries, we require that the cycles be contained in the tight closure of the boundaries. If this condition occurs in a complex, Hochster and Huneke say that the complex has *phantom homology* at that spot. For a thorough introduction to phantom homology and phantom acyclicity. see [HH1, Section 9] and [HH4].

The phantom acyclicity criterion given below can be thought of as a tight closure analogue of the Buchsbaum-Eisenbud acyclicity criterion, which gives sufficient rank and depth conditions for a complex to be acyclic. In the phantom case, rank and height conditions are given to ensure that a complex has phantom homology.

Theorem 2.2.11 (Phantom Acyclicity Criterion, [HH1])**.** *Let R be a positive characteristic, reduced Noetherian ring that is also locally equidimensional and a homomorphic image of a Cohen-Macaulay ring. Let $G_\bullet$ be a finite free complex over R:*

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0.$$

Let α_i be the matrix map $G_i \to G_{i-1}$, and let r_i be the determinantal rank of α_i, for $1 \le i \le n$. Let b_i be the free rank of G_i, for $0 \le i \le n$. Denote the ideal generated by the size r minors of a matrix map α by $I_r(\alpha)$. If $b_i = r_i + r_{i-1}$ and $\operatorname{ht} I_{r_i}(\alpha_i) \ge i$, for all $1 \le i \le n$, then $Z_i \subseteq (B_i)^*_{G_i}$, where B_i is the image of α_{i+1} and Z_i is the kernel of α_i. In other words, if the previous rank and height conditions are satisfied, then the cycles are contained in the tight closure of the boundaries.

Hochster and Huneke prove the following result, which is often referred to as "colon-capturing."

Theorem 2.2.12 (Theorem 7.15, [HH1])**.** *Let R be a local positive characteristic Noetherian ring that is module-finite and torsion-free over a regular ring A or is locally equidimensional and a homomorphic image of a Cohen-Macaulay ring. If $x_1, \ldots, x_n$ is a system of parameters for R, then*

$$(x_1, \ldots, x_k)R :_R x_{k+1} \subseteq (x_1, \ldots, x_k)^*_R,$$

for all $0 \le k \le n-1$.

In a sense, colon-capturing shows that the tight closure of parameter ideals measures the obstruction for such a ring R to be Cohen-Macaulay. In the following sections, we will see that tight closure, parameter ideals, and the Cohen-Macaulay property have many interconnections.

2.3 Big Cohen-Macaulay Algebras

A *big Cohen-Macaulay module* M over a local Noetherian ring (R, m) is an R-module such that every system of parameters for R is a regular sequence on M. If $M = B$ is an R-algebra, then B is called a *big Cohen-Macaulay algebra*. The terminology "big" refers to the fact that M is not necessarily a finitely generated R-module. Also note that what we refer to as big Cohen-Macaulay algebras are called *balanced* by some authors, who call an algebra B big Cohen-Macaulay if a single system of parameters for R is a regular sequence on B. Recall that part of the definition of a regular sequence $x_1, \ldots, x_d$ requires that $(x_1, \ldots, x_d)M \neq M$. Since we will require this condition for every system of parameters of R, we can equivalently require that $mM \neq M$. It will also be useful to say that a partial system of parameters of (R, m) is a *possibly improper regular sequence* on M if all relations on the parameters are trivial, but $mM \neq M$ does not necessarily hold. Similarly, we can have *possibly improper big Cohen-Macaulay algebras.*

The question "When do big Cohen-Macaulay algebras exist?" has important applications to the class of problems in commutative algebra that is often referred to as the "local homological conjectures." This class of problems includes the direct summand conjecture and the monomial conjecture, and contains a number of interconnected statements. The truth of many of them is implied by a *weakly functorial existence* of big Cohen-Macaulay algebras over complete local domains. Hochster and Huneke use this terminology to mean that for any local rings R and S with a "reasonably good" local map $R \to S$, there exist big Cohen-Macaulay algebras $\mathscr{B}(R)$

and $\mathscr{B}(S)$ over R and S, respectively, such that

(2.3.1)
$$\begin{tikzcd} \mathscr{B}(R) \arrow[r] & \mathscr{B}(S) \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

is a commutative diagram. We will make the notion of "reasonably good" precise for Theorem 2.3.5.

Over regular rings, the search for big Cohen-Macaulay modules and algebras is rather easy given the following observation, a short proof of which may be found in [HH2, p.77].

Proposition 2.3.2. *Let R be a regular Noetherian ring, and let M be an R-module. Then M is a big Cohen-Macaulay module over R if and only if M is faithfully flat over R.*

The first significant existence proof of big Cohen-Macaulay algebras came from the celebrated theorem of Hochster and Huneke:

Theorem 2.3.3 (Theorem 5.15, [HH2])**.** *Let R be an excellent local domain, and let R^+ denote the integral closure of R in an algebraic closure of its fraction field. Then R^+ is a big Cohen-Macaulay R-algebra.*

The ring R^+ above is called the *absolute integral closure of* R and is not Noetherian in general. Since local maps $R \to S$ between excellent local domains extend to maps $R^+ \to S^+$, the theorem above yields a weakly functorial existence of big Cohen-Macaulay algebras in positive characteristic.

Hochster and Huneke also provide a graded result of the above theorem. In the graded case, they work with graded subrings of R^+. Specifically, if R is an $\mathbb{N}$-graded domain, then $R^{+\mathrm{GR}}$ denotes a maximal direct limit of module-finite, $\mathbb{Q}_{\geq 0}$-graded extension domains of R. For the construction and properties of this ring; see [HH3, Lemma 4.1]. This result also shows that there is an $\mathbb{N}$-graded direct summand of $R^{+\mathrm{GR}}$, which is denoted $R^{+\mathrm{gr}}$. Neither of these rings are Noetherian in general.

Theorem 2.3.4 (Theorem 5.15, [HH2]). *If R is an $\mathbb{N}$-graded domain with $R_0 = K$ and R a finitely generated K-algebra, then $R^{+\mathrm{GR}}$ and $R^{+\mathrm{gr}}$ are both graded big Cohen-Macaulay R-algebras in the sense that every homogeneous system of parameters of R is a regular sequence on $R^{+\mathrm{GR}}$ and $R^{+\mathrm{gr}}$.*

Using their result for R^+, Hochster and Huneke were also able to establish a weakly functorial existence of big Cohen-Macaulay algebras over all equicharacteristic local rings. The term *permissible* used in the following theorem refers to a map $R \to S$ such that every minimal prime Q of $\widehat{S}$, with $\dim \widehat{S}/Q = \dim \widehat{S}$, lies over a prime P of $\widehat{R}$ that contains a minimal prime $\mathfrak{p}$ of $\widehat{R}$ satisfying $\dim \widehat{R}/\mathfrak{p} = \dim \widehat{R}$.

Theorem 2.3.5 (Theorem 3.9, [HH7]). *We may assign to every equicharacteristic local ring R a big Cohen-Macaulay R-algebra $\mathscr{B}(R)$ in such a way that if $R \to S$ is a permissible local homomorphism of equicharacteristic local rings, then we obtain a homomorphism $\mathscr{B}(R) \to \mathscr{B}(S)$ and a commutative diagram as in (2.3.1).*

A key tool in the proof of this result is the construction of big Cohen-Macaulay algebras using *algebra modifications*. Since we will make great use of algebra modifications in this thesis, it will be helpful to review some definitions and useful properties now.

Given a local Noetherian ring R, an R-algebra S, and a relation $sx_{k+1} = \sum_{i=1}^{k} x_i s_i$ in S, where $x_1, \ldots, x_{k+1}$ is part of a system of parameters of R, the S-algebra

$$T := \frac{S[U_1, \ldots, U_k]}{s - \sum_{i=1}^{k} x_i U_i} \tag{2.3.6}$$

is called an *algebra modification of S over R*.

Instead of constructing an algebra modification with respect to a single relation on a single system of parameters from R, one can also create an S-algebra $\mathrm{Mod}(S/R) = \mathrm{Mod}_1(S/R)$ by adding infinitely many indeterminates and killing the appropriate relations (as above) so that *every* relation in S on a partial system of parameters from R is trivialized in $\mathrm{Mod}_1(S/R)$. Now inductively define $\mathrm{Mod}_n(S/R) = \mathrm{Mod}(\mathrm{Mod}_{n-1}(S/R))$ and then define $\mathrm{Mod}_\infty(S/R)$ as the direct limit of the $\mathrm{Mod}_n(S/R)$. The utility of this construction is that we have formally killed all possible relations on systems of parameters from R and done so in a way that is universal in the following sense.

Proposition 2.3.7 (Proposition 3.3b, [HH7]). *Let S be an algebra over the local Noetherian ring R. Then $\mathrm{Mod}_\infty(S/R)$ is a possibly improper big Cohen-Macaulay R-algebra. It is a proper big Cohen-Macaulay algebra if and only if S maps to some big Cohen-Macaulay R-algebra.*

While $\mathrm{Mod}_\infty(S/R)$ is a rather large and cumbersome object, we can study it in terms of finite sequences of algebra modifications. Given a Noetherian local ring (R, m) and an R-algebra S, we defined an algebra modification T of S in (2.3.6). If we set $S^{(0)} := S$ and then inductively define $S^{(i+1)}$ to be an algebra modification of $S^{(i)}$ over R, we can obtain a finite *sequence of algebra modifications*

$$S = S^{(0)} \to S^{(1)} \to \cdots \to S^{(h)},$$

for any $h \in \mathbb{N}$. We call such a sequence *bad* if $mS^{(h)} = S^{(h)}$. The utility of these sequences can be found in the following proposition, which we will use frequently in the later chapters.

Proposition 2.3.8 (Proposition 3.7, [HH7]). *Let R be a local Noetherian ring, and let S be an R-algebra. $\mathrm{Mod}_\infty(S/R)$ is a proper big Cohen-Macaulay R-algebra if and* *only if no finite sequence of algebra modifications is bad.*

2.4 Plus Closure

Beyond being a big Cohen-Macaulay algebra in positive characteristic, the ring R^+ also has connections to tight closure theory. In fact, it has long been speculated that the tight closure of an ideal is just the contracted-expansion from R^+. This notion led to the definition of *plus closure*.

Definition 2.4.1. Given an excellent, local domain R of positive characteristic, and finitely generated modules $N \subseteq M$, we define the *plus closure* of N in M to be

$$N_M^+ := \{u \in M \mid 1 \otimes u \in \operatorname{Im}(R^+ \otimes_R N \to R^+ \otimes_R M)\}.$$

In the case the $M = R$ and $N = I$, then $I^+ = IR^+ \cap R$.

As with R^+, there is a closure operation associated to $R^{+\mathrm{gr}}$ and $R^{+\mathrm{GR}}$, which we define below. Since $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$ as a $R^{+\mathrm{gr}}$-module, the two rings yield equivalent closure operations.

Definition 2.4.2. Let R be an $\mathbb{N}$-graded Noetherian domain. Let $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$, and let $N \subseteq M$ be finitely generated R-modules with $u \in M$. Then $u \in N_M^{+\mathrm{gr}}$, the *graded-plus closure* of N in M, if $1 \otimes u \in \operatorname{Im}(S \otimes N \to S \otimes M)$. For the ideal case, $I^{+\mathrm{gr}} = IR^{+\mathrm{gr}} \cap R = IR^{+\mathrm{GR}} \cap R$.

It is straightforward to show that $N_M^F \subseteq N_M^+ \subseteq N_M^*$, for all finitely generated modules, and that $N_M^F \subseteq N_M^{+\mathrm{gr}} \subseteq N_M^+ \subseteq N_M^*$ in the graded case.

Since $U^{-1}(R^+) \cong (U^{-1}R)^+$, for any multiplicative set U in R, the computation of plus closure commutes with localization. As mentioned earlier, it is hoped that tight closure in positive characteristic is just plus closure. Not only would this equality simplify the computation of tight closure, it would also settle the long standing localization question by proving that tight closure commutes with localization. One of the most important results in this direction comes from K.E. Smith.

Theorem 2.4.3 (Theorem 5.1, [Sm1]). *Let R be an excellent local Noetherian domain of characteristic $p > 0$. Then $(x_1, \ldots, x_k)^* = (x_1, \ldots, x_k)^+$, where $x_1, \ldots, x_k$ is part of a system of parameters in R.*

In [Ab2, Theorem 3.1], I. Aberbach extends Smith's result to include all finitely generated modules $N \subseteq M$ over an excellent local domain R of positive characteristic such that M/N has *finite phantom projective dimension*, which is a tight closure analogue of the usual notion of finite projective dimension. See [Ab] for more detail.

Smith also produced a proof in the graded case.

Theorem 2.4.4 (Theorem 1, [Sm2]). *Let $I = (x_1, \ldots, x_k)$ be an ideal generated by part of a homogeneous system of parameters for a Noetherian $\mathbb{N}$-graded domain R with R_0 a field of positive characteristic. Then $I^* = IR^{+\mathrm{gr}} \cap R = IR^{+\mathrm{GR}} \cap R$.*

Recently, H. Brenner has developed new results in dimension 2 that show that tight closure and graded-plus closure are equivalent for homogeneous ideals in certain graded rings. A primary ingredient in his work has been an interesting correspondence between tight closure membership and the properties of certain projective bundles; see [Br1]. Using this bundle notion of tight closure, Brenner has successfully applied geometric theorems and techniques (such as Atiyah's classification of vector bundles on an elliptic curve and Harder-Narasimhan filtrations of Frobenius pull-backs of locally free sheaves) to arrive at the following results.

Theorem 2.4.5 (Theorem 4.3, [Br2]). *Let K be an algebraically closed field of positive characteristic, and let R be the homogeneous coordinate ring of an elliptic curve (i.e., R is a standard graded normal K-algebra of dimension 2 with $\dim_K[H^2_m(R)]_0 =$*

1, *where* m *is the homogeneous maximal ideal of* R*). Let* I *be an* m*-primary graded ideal in* R*. Then* $I^{+\mathrm{gr}} = I^{+} = I^{*}$*.*

Remark 2.4.6. For example, the result above applies when $R = K[x, y, z]/(F)$ is normal, where F is homogeneous of degree 3.

Theorem 2.4.7 (Theorem 4.2, [Br3])**.** *Let* K *be the algebraic closure of a finite field. Let* R *denote an* $\mathbb{N}$*-graded 2-dimensional domain of finite type over* K*. Then for every homogeneous ideal* I*, we have* $I^{+\mathrm{gr}} = I^{+} = I^{*}$*.*

Given an elliptic curve X over a positive characteristic p field, there exists a Frobenius map $F : X \to X$ which is the identity on the set X and the p^{th} power map on the structure sheaf $\mathcal{O}_X$. The map F induces a map $F^{*} : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ on cohomology. One says that X has *Hasse invariant* 0 if $F^{*} = 0$. We refer the reader to [Ha, pp. 332–335] for further details.

In the case of Hasse invariant 0, Brenner showed that tight closure is the same as Frobenius closure.

Theorem 2.4.8 (Remark 4.4, [Br2])**.** *If* R *is the homogeneous coordinate ring of an elliptic curve of positive characteristic* p *with Hasse invariant 0 defined over an algebraically closed field, then* $I^{*} = I^{F}$ *for all* m*-primary graded ideals of* R*, where* m *is the homogeneous maximal ideal of* R*.*

2.5 Solid Algebras and Solid Closure

Hochster introduced the notion of *solid* modules and algebras in [Ho3] as the basis for defining a characteristic free closure operation, which he called *solid closure*. The hope was that solid closure might be a good enough operation to imply the existence of big Cohen-Macaulay algebras in mixed characteristic settings or to help answer some of the homological conjectures in mixed characteristic. While it turns out that solid closure is equivalent to tight closure in positive characteristic, at least over complete local domains, solid closure is too large in equal characteristic 0, as shown by an example of P. Roberts (see [Ro]). The situation in mixed characteristic is still a mystery.

Definition 2.5.1. If R is a Noetherian domain, then an R-module M is *solid* if $\mathrm{Hom}_R(M, R) \neq 0$. If $M = S$ is an R-algebra, then S is solid over R if it is solid as an R-module.

We will present just a few of the properties of solid modules and algebras as these properties will be helpful in the following chapters. All of these properties can be found in [Ho3, Section 2].

Proposition 2.5.2. *Let R be a Noetherian domain.*

(a) *If M and N are solid R-modules, then $M \otimes_R N$ is solid.*

(b) *If S is a solid R-algebra, then there exists an R-linear map $\alpha : S \to R$ such that $\alpha(1) \neq 0$ in R.*

(c) *Let S be a module-finite domain extension of R, and let M be an S-module. Then M is solid over S if and only if M is solid over R.*

(d) ***(local cohomology criterion)*** *If (R, m) is a complete local domain of Krull dimension n, then an R-module M is solid if and only if $H^n_m(M) \neq 0$.*

(e) ***(persistence of solidity)*** *Let $R \to S$ be any map of Noetherian domains. If M is a solid R-module, then $S \otimes_R M$ is a solid S-module.*

If R is a Noetherian ring, then a *complete local domain of R* is an R-algebra T obtained by completing the localization of R at a maximal ideal and then killing a minimal prime. Using this notion, Hochster defined the following closure operation.

Definition 2.5.3. Let R be a Noetherian ring, and let $N \subseteq M$ be finitely generated R-modules. If R is a complete local domain, then the *solid closure* of N in M is

$$N_M^{\bigstar} := \{u \in M \mid 1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M), \text{ for some solid } R\text{-algebra } S\}.$$

For a general Noetherian ring R, an element $u \in M$ is in $N_M^{\bigstar}$ if for every complete local domain T of R, the image $1 \otimes u$ of u in $T \otimes_R M$ is in the solid closure of $\operatorname{Im}(T \otimes_R N \to T \otimes_R M)$ in $T \otimes_R M$ over T.

As mentioned above, solid closure is equivalent to tight closure in positive characteristic, at least when the ring contains a completely stable test element.

Theorem 2.5.4 (Theorem 8.6, [Ho3])**.** *Let R be a Noetherian ring of positive characteristic, and let $N \subseteq M$ be finitely generated R-modules.*

(a) $N_M^* \subseteq N_M^{\bigstar}$.

(b) *If R has a completely stable test element, e.g., R is a complete local domain, then $N_M^* = N_M^{\bigstar}$.*

The connection between solid closure and tight closure also leads to more connections between tight closure and big Cohen-Macaulay algebras and is also a factor in the work of Brenner mentioned in the last section, where the correspondence between tight closure and projective bundles uses solid closure as an intermediary. As for big Cohen-Macaulay algebras, we point out the following two facts and will discuss these connections further in following chapters, including whether the next statement has a converse in positive characteristic.

Theorem 2.5.5 (Corollary 10.6, [Ho3])**.** *Let R be a complete local domain. An R-algebra that has an R-algebra map to a big Cohen-Macaulay algebra over R is solid.*

Theorem 2.5.6 (Theorem 11.1, [Ho3])**.** *Let R be a complete local domain of positive characteristic, and let $N \subseteq M$ be finitely generated R-modules. If $u \in M$, then $u \in N_M^* = N_M^{\bigstar}$ if and only if there exists a big Cohen-Macaulay R-algebra B such that $1 \otimes u \in \mathrm{Im}(B \otimes_R N \to B \otimes_R M)$.*

CHAPTER 3

Tight Closure of Finite Length Modules in Graded Rings

In this chapter, we will investigate some conditions in which tight closure and plus closure (or even Frobenius closure) are the same. Although our main result will not depend upon dimension, the primary applications are based on results known in dimension 2, The cubical cone

$$R = K[x, y, z]/(x^3 + y^3 + z^3)$$

will be a primary example, where our results about Frobenius closure will apply when the characteristic of K is congruent to $2 \pmod 3$.

In the next section we look at how the equivalence of tight closure and plus closure (or Frobenius closure) in the homogeneous m-coprimary case implies the same closure equivalence in the non-homogeneous m-coprimary case. The second section shows some consequences of the first section, where we will apply Theorem 3.1.6 to the recent work of H. Brenner in [Br2] and [Br3] (see Theorems 2.4.5, 2.4.7, and 2.4.8). We also demonstrate a connection between tight closure and the m-adic closure of modules extended to R^+ (or R^∞).

The third section details some work concerning the injective hull of the residue field in R^∞, R^+, and $R^{+\mathrm{GR}}$. Our results show that unlike the Noetherian case, these injective hulls contain elements that are not killed by any power of the maximal ideal of R. This fact is relevant because it presents an obstruction to one possible method of extending the result of Theorem 3.1.6 to all ideals.

3.1 New Cases Where Tight Closure is Plus Closure

Before proving the main result of this section, Theorem 3.1.6, we need to establish two lemmas and some notation. If S is any $\mathbb{Q}$-graded (not necessarily Noetherian) ring, then for any $n \in \mathbb{Q}$ let $S_{\geq n} = \bigoplus_{i \geq n} S_i$. Similarly define $S_{>n}$. We will say that an $\mathbb{N}$-graded ring R is a *standard graded R_0-algebra* if R is finitely generated over R_0 by elements of degree 1. For the rest of the section, let $m = \bigoplus_{i>0} R_i$, the homogeneous maximal ideal of R.

Lemma 3.1.1. *Let R be a reduced standard graded K-algebra of positive characteristic p, and let $S = R^\infty$. Then there exists $c \in \mathbb{N}$ such that $S_{\geq n+c} \subseteq m^n S$ for any $n \geq 1$. As a consequence, $[S/m^n S]_j = 0$ for all $j \gg 0$.*

Proof. Let m be generated by $x_1, \ldots, x_\mu$, each of degree 1. Put $c = \mu - 1$ (if $\mu = 0$, i.e., $R = K$, put $c = 0$). Since $S_{>0} = \cup_q m^{1/q}$, if $f \in S$ is homogeneous of degree at least $n+c$, then f is a sum of terms $cx_1^{\alpha_1} \cdots x_\mu^{\alpha_\mu}$ such that $c \in S_0$ and $\sum \alpha_i \geq n+c$. If we write $\alpha_i = [\alpha_i] + r_i$, where $0 \leq r_i < 1$ for all i, then

$$\sum [\alpha_i] = \sum \alpha_i - \sum r_i \geq n + c - \sum r_i > n + c - \mu.$$

Therefore, $\sum [\alpha_i] \geq n + c - \mu + 1 = n$, and so $f \in m^n S$. The second claim now follows using $j \geq n + c$. $\square$

For the other lemma we will need a graded-plus closure version of the Briançon-Skoda Theorem. The original tight closure generalization (Theorem 2.2.4) can be found in [HH1]. Hochster and Huneke also strengthened this result to a version for plus closure.

Theorem 3.1.2 (Theorem 7.1, [HH7]). *Let I be an ideal of a Noetherian domain R of characteristic $p > 0$ generated by at most d elements, let $k \in \mathbb{N}$, and let $u \in \overline{I^{d+k}}$. Then $u \in I^{k+1}R^+ \cap R$.*

We will adapt their proof to obtain a graded-plus closure version of the Briançon-Skoda Theorem. In order to simplify the proof we will also use Smith's result for tight closure of homogeneous parameter ideals (see Theorem 2.4.4).

Theorem 3.1.3. *Let R be a positively graded Noetherian domain of positive characteristic. Let I be a homogeneous ideal generated by at most d homogeneous elements, let $k \in \mathbb{N}$, and let $u \in \overline{I^{d+k}}$ with u homogeneous. Then $u \in I^{k+1}S \cap R$, where $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$.*

Proof. As $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$, it is sufficient to assume that $S = R^{+\mathrm{GR}}$. Let $A \to R$ be a degree-preserving map of positively graded Noetherian domains such that J is a homogeneous ideal of A with at most d generators, and $I = JR$. Let $v \in \overline{J^{d+k}}$ be a homogeneous element of A such that $v \mapsto u$ in R. Suppose that $v \in J^{k+1}A^{+\mathrm{GR}}$. Since the map $A \to R$ extends to $A^+ \to R^+$, we can restrict this map to obtain $A^{+\mathrm{GR}} \to R^{+\mathrm{GR}}$ (The homogeneous monic equation satisfied by an element a of $A^{+\mathrm{GR}}$ maps to a homogeneous monic equation over R satisfied by the image of a.) Therefore, $u \in I^{k+1}R^{+\mathrm{GR}}$.

Now, let $K = \mathbb{Z}/p\mathbb{Z}$. Since u is integral over I^{d+k} and homogeneous, it satisfies a homogeneous monic polynomial $z^n + r_1 z^{n-1} + \cdots + r_n = 0$, where $\deg z = \deg u$, $\deg r_j = j \deg u$, $r_j \in (I^{d+k})^j$, and (without loss of generality) $r_n \neq 0$.

Each r_j can be written as a homogeneous R-linear combination of monomials $a_1^{\nu_1} \cdots a_d^{\nu_d}$ in the generators $a_1, \ldots, a_d$ of I, where $\nu_1 + \cdots + \nu_d = (d+k)j$. Thus, the coefficient on the monomial $a_1^{\nu_1} \cdots a_d^{\nu_d}$ is zero or has

$$\text{degree } = \deg r_j - (\nu_1 \deg a_1 + \cdots + \nu_d \deg a_d)$$

since R is positively graded. Without loss of generality, we may order the generators of I so that $\deg a_1 \leq \cdots \leq \deg a_d$. Then $r_n \neq 0$ implies that $\deg a_1^{(d+k)n} \leq \deg r_n$. If not, then $\deg r_n < \nu_1 \deg a_1 + \cdots + \nu_d \deg a_d$, for all ν_i such that $\nu_1 + \cdots + \nu_d = (d+k)n$, and so the coefficient on every monomial in the expansion of r_n must be zero, a contradiction.

Let $x_1, \ldots, x_d$ be indeterminates over K with $\deg x_i = \deg a_i$. For every monomial $\mu = x_1^{\nu_1} \cdots x_d^{\nu_d}$, where $\nu_1 + \cdots + \nu_d = (d+k)j$, for $1 \leq j \leq n$, let y_μ be an indeterminate with $\deg y_\mu = \deg r_j - (\nu_1 \deg x_1 + \cdots + \nu_d \deg x_d)$. Let

$$F(\mathbf{x}, \mathbf{y}, z) = z^n + \sum_{j=1}^{n} \left(\sum_{\mu \in C_j} y_\mu \mu \right) z^{n-j},$$

where $\mathbf{x} = x_1, \ldots, x_d$, $\mathbf{y} = \{y_\mu \mid \deg y_\mu \geq 0\}$, and

$$C_j = \{\mu = x_1^{\nu_1} \cdots x_d^{\nu_d} \mid \nu_1 + \cdots + \nu_d = (d+k)j\}.$$

Then F is homogeneous of degree $n \deg z = n \deg u$ as

$$\deg(y_\mu \mu) z^{n-j} = \deg r_j + (n-j) \deg z = j \deg u + n \deg z - j \deg z = n \deg z.$$

Therefore, $K[\mathbf{x}, \mathbf{y}, z]$ is a positively graded Noetherian ring, and $K[\mathbf{x}, \mathbf{y}] \to R$, given by $x_i \mapsto a_i$ and y_μ mapping to the coefficient of $a_1^{\nu_1} \cdots a_d^{\nu_d}$, is a degree-preserving map. Moreover, the composite map $K[\mathbf{x}, \mathbf{y}, z] \to R[z] \to R$, where $z \mapsto u$, sends $F(\mathbf{x}, \mathbf{y}, z) \mapsto z^n + r_1 z^{n-1} + \cdots + r_n \mapsto 0$. (Since R is positively graded, the coefficient on $a_1^{\nu_1} \cdots a_d^{\nu_d}$ is 0 if $\deg r_j < \nu_1 \deg a_1 + \cdots + \nu_d \deg a_d$ so that we did not need a $y_\mu \mu$ term in F when $\deg y_\mu < 0$.)

Let $A = K[\mathbf{x}, \mathbf{y}, z]/F(\mathbf{x}, \mathbf{y}, z)$, and $J = (\mathbf{x})A$. This is a positively graded Noetherian ring of positive characteristic, J is a homogeneous ideal generated by at most d homogeneous elements, and z is homogeneous such that $z \in \overline{J^{d+k}}$. It is clear from the construction of A that $A \to R$ is a degree-preserving map. To see that A is also a domain, we will show that F is irreducible. Indeed, let $N = (d+k)n$, and let μ be the monomial x_1^N that occurs when $j = n$ in the summation for F. As we noted earlier, $r_n \neq 0$ implies that

$$\deg y_\mu = \deg r_n - N \deg x_1 = \deg r_n - N \deg a_1 \geq 0,$$

and so F is linear in y_μ with coefficient x_1^N on y_μ and a relatively prime constant term containing z^n.

We may, therefore, assume that R is A, and I is J. Since R is a positively graded, finitely generated K-algebra, we may regrade if necessary so that R is $\mathbb{N}$-graded without changing $R^{+\mathrm{GR}}$. Since $R/(\mathbf{x}) \cong K[\mathbf{y}, z]/z^n$, the sequence $x_1, \ldots, x_d$ forms part of a homogeneous system of parameters. We can now apply Smith's Theorem 2.4.4 to the ring R and ideal I to see that $I^* = IR^{+\mathrm{gr}} \cap R = IR^{+\mathrm{GR}} \cap R$. By Theorem 3.1.2, we are done. $\square$

Lemma 3.1.4. *Let R be a standard graded K-algebra domain of characteristic $p > 0$, and let $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$. Then there exists $c \in \mathbb{N}$ such that $S_{\geq n+c} \subseteq m^n S$, for any $n \geq 1$. Moreover, $[S/m^n S]_j = 0$ for all $j \gg 0$.*

Proof. Let m be generated by μ elements. Let $c = \mu - 1$ (if $\mu = 0$, let $c = 0$), and let $f \in S$ be homogeneous of degree $D \geq n + c$. Then f satisfies a monic polynomial equation $f^t + r_1 f^{t-1} + \cdots + r_t = 0$ such that r_i is homogeneous of degree iD in R or $r_i = 0$ if $iD \notin \mathbb{N}$. Therefore, $r_i \in m^{i(n+c)} = (m^{n+c})^i$ for all i as m is generated in degree 1. Since $f \in S$, there exists a positively graded module-finite extension domain T of R such that $f \in T$. Thus, $f \in \overline{(mT)^{n+c}} = \overline{(mT)^{\mu+n-1}}$. By Theorem 3.1.3, $f \in m^n T^{+\mathrm{GR}}$, but $T^{+\mathrm{GR}} = R^{+\mathrm{GR}}$, and so $f \in m^n R^{+\mathrm{GR}}$. Since $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$, we also have $f \in m^n R^{+\mathrm{gr}}$. The second claim follows using $j \geq n + c$. □

Our main result will depend upon showing that $\mathrm{Hom}_K(S/m^n S, K)$ is $\mathbb{Z}$-graded as an R-module when S is $R^{+\mathrm{GR}}$, $R^{+\mathrm{gr}}$, or R^{∞}.

Proposition 3.1.5. *Let R be a standard graded K-algebra of characteristic $p > 0$. Suppose R is reduced (respectively, a domain). Let $S = R^{\infty}$ (resp., $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$). Then for any $n \geq 1$,* $\mathrm{Hom}_K(S/m^n S, K)$ *is a $\mathbb{Z}$-graded R-module.*

Proof. S has a natural $\mathbb{N}[1/p]$-grading (resp., $\mathbb{Q}_{\geq 0}$-grading or $\mathbb{N}$-grading) induced by the grading on R. Thus, $S/m^n S$ is also graded as $m^n S$ is a homogeneous ideal. Let W_j be the K-span of all homogeneous elements of degree δ such that $j - 1 < \delta \leq j$. This gives $S/m^n S$ an $\mathbb{N}$-grading as an R-module, where $W_j = 0$ for all $j < 0$ and $j \gg 0$ by Lemma 3.1.1 (resp., Lemma 3.1.4).

In $\mathrm{Hom}_K(S/m^n S, K)$, let V_{-j} be the K-span of all functionals ϕ such that $\phi(W_i)$ is not 0 if and only if $i = j$. If $r \in R$ is homogeneous of degree d, and $\phi \in V_{-j}$, then $r\phi(W_i) = \phi(rW_i) \subseteq \phi(W_{i+d})$ which is nonzero if and only if $i = -d + j$. Thus, $R_d V_{-j} \subseteq V_{-j+d}$. It is clear that the intersection of any V_{-j} with the sum of the others is trivial and that $\sum_j V_{-j} \subseteq \mathrm{Hom}_K(S/m^n S, K)$. Now, if $\psi \in \mathrm{Hom}_K(S/m^n S, K)$, and s has homogeneous components s_i, then let $\psi_{-j}(s) = \psi(s_j)$ so that $\psi_{-j} \in V_{-j}$. Then $\psi = \sum_j \psi_{-j}$, where the sum is finite because W_i is nonzero for only finitely many integers. Therefore the V_{-j} give a $\mathbb{Z}$-grading on $\mathrm{Hom}_K(S/m^n S, K)$ as an R-module. □

We are now ready to present the main result of this chapter. The method of the proof will be to show that if M is an m-coprimary module containing an element $u \in 0^*_M \setminus 0^{+\mathrm{GR}}_M$, then M can be mapped to a finitely generated graded m-coprimary R-module where the image of u is not in the plus closure of 0 in this new module.

Theorem 3.1.6. *Let R be a standard graded K-algebra of characteristic $p > 0$. Suppose that R is reduced (respectively, a domain), and K is perfect (resp., algebraically closed). If $N^*_M = N^F_M$ (resp., $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$) for all finitely generated graded R-modules $N \subseteq M$ such that M/N is m-coprimary, then the same is true for all finitely generated modules $N \subseteq M$ such that M/N is m-coprimary.*

Proof. Let $S = R^\infty$ (resp., $S = R^{+\mathrm{GR}}$). It suffices to show that $0^*_M \subseteq 0^F_M$ (resp., $0^*_M \subseteq 0^{+\mathrm{gr}}_M$) when M is m-coprimary. Suppose that $u \in 0^*_M \setminus 0^F_M$ (resp., $u \in 0^*_M \setminus 0^{+\mathrm{gr}}_M$). Since $u \notin 0^F_M$ (resp., $u \notin 0^{+\mathrm{gr}}_M$), by Lemma 2.1.2 (resp., Definition 2.4.2) $1 \otimes u \neq 0$ via the map $M \to S \otimes M$. This implies that there is a surjection of $S(1 \otimes u)$ onto K sending $1 \otimes u$ to $1 \in K$, since the residue field of S is K.

Since $\mathrm{Hom}_K(S, K)$ is an injective S-module and we have a map $K \to \mathrm{Hom}_K(S, K)$ that sends 1 to the functional that takes $s \in S$ to s modulo m_S, we see that

$$M \to S \otimes M \to \mathrm{Hom}_K(S, K)$$

is a map such that u is not in the kernel. Since M is m-coprimary, there exists an n such that $m^n M = 0$. Hence, the image of M under the composite map above lies in the annihilator of m^n in $\mathrm{Hom}_K(S, K)$, which is isomorphic to $\mathrm{Hom}_K(S/m^n S, K)$.

By Proposition 3.1.5, $\mathrm{Hom}_K(S/m^n S, K)$ is a $\mathbb{Z}$-graded R-module. Let M' be the R-submodule of $\mathrm{Hom}_K(S/m^n S, K)$ generated by the homogeneous components of the generators of the image of M. Therefore, M' is a finitely generated graded R-module, and since m^n kills M and is a homogeneous ideal, it kills M' as well. Thus, M' is also m-coprimary as an R-module.

Let $\tilde{u}$ be the image of u in M', which we know is nonzero. As $u \in 0^*_M$, we also have that $\tilde{u} \in 0^*_{M'}$. By our hypothesis, $\tilde{u} \in 0^F_{M'}$ (resp., $\tilde{u} \in 0^{+\text{gr}}_{M'}$) since M' is graded and m-coprimary. Therefore, $1 \otimes \tilde{u} = 0$ in $S \otimes M'$. Since $\text{Hom}_K(S/m^nS, K)$ is an S-module, the inclusion map $M' \hookrightarrow \text{Hom}_K(S/m^nS, K)$ factors through the map $M' \to S \otimes M'$, by the universal property of base change. Thus, the fact that $1 \otimes \tilde{u} = 0$ in $S \otimes M'$ implies that the image of $\tilde{u}$ is 0 in $\text{Hom}_K(S/m^nS, K)$, a contradiction. $\square$

3.2 Consequences

When R is the homogeneous coordinate ring of an elliptic curve of positive characteristic p over an algebraically closed field (i.e., R is a standard graded normal K-algebra of dimension 2 with $\dim_K [H^2_m(R)]_0 = 1$, where m is the homogeneous maximal ideal of R), or when R is an $\mathbb{N}$-graded 2-dimensional domain of finite type over K, where K is the algebraic closure of a finite field, we would like to apply Theorem 3.1.6 to the results of Brenner.

In [Br2] and [Br3], Brenner shows that the tight closure of a primary homogeneous ideal is the same as its graded-plus closure. Brenner has observed in correspondence that it is straightforward to generalize Theorems 2.4.5 and 2.4.7 to include finitely generated m-coprimary R-modules. The argument is lengthy, like the one for ideals, but the changes are routine. (The main idea is to replace the syzygy bundle constructed from homogeneous generators of an m-primary ideal with a syzygy bundle constructed from homogeneous generators of an m-coprimary submodule N of a graded module M. Once one has made the necessary alterations to the results in [Br1, Section 3], all of the relevant proofs in [Br2] and [Br3] follow seamlessly as they only rely on the aforementioned results and theorems whose hypotheses only require locally free sheaves of arbitrary rank, which we obtain in the ideal and module cases.)

With some degree of caution, we state this generalization as the following theorem.

Theorem 3.2.1 (Brenner). *Let R be a positive characteristic ring. Further, let R be the homogeneous coordinate ring of an elliptic curve over an algebraically closed field K, or let R be any 2-dimensional standard graded K-algebra, where K is the algebraic closure of a finite field. Let $N \subseteq M$ be finitely generated graded R-modules such that M/N is m-coprimary, where m is the homogeneous maximal ideal of R. Then $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$.*

Remark 3.2.2. For example, R is the homogeneous coordinate ring of such an elliptic curve when $R = K[x, y, z]/(F)$, where R is normal and F is homogeneous of degree 3. Therefore, the result above applies to the cubical cone $R = K[x, y, z]/(x^3 + y^3 + z^3)$.

This result together with Theorem 3.1.6 yields the following corollary.

Corollary 3.2.3. *With R as above, $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$ for all finitely generated R-modules such that M/N is m-coprimary.*

Further, if Proj R is an elliptic curve with Hasse invariant 0 (see Section 2.4 or [Ha, pp. 332-335]), then Brenner's Theorem 2.4.8 says that the tight closure of a primary homogeneous ideal is the same as its Frobenius closure. For example, this is the case for the cubical cone $R = K[x, y, z]/(x^3 + y^3 + z^3)$, when the characteristic of K is congruent to $2 \pmod 3$ (as implied by [Ha, Proposition 4.21]). Again, Brenner's result can be generalized to include finitely generated homogeneous modules $N \subseteq M$ with m-coprimary quotients. This fact can then be paired with Theorem 3.1.6 to give:

Corollary 3.2.4. *If R is the homogeneous coordinate ring of an elliptic curve of positive characteristic p with Hasse invariant 0 defined over an algebraically closed* *field, then* $N_M^* = N_M^F$ *for all finitely generated* R*-modules such that* M/N *is* m*-coprimary.*

For a Noetherian ring R with a maximal ideal m, given the equivalence of tight closure and plus closure (respectively, graded-plus closure or Frobenius closure) in the m-coprimary case, we can present a characterization of tight closure in the localization of R at m in terms of the m-adic closure of modules inside R^+ (resp., $R^{+\mathrm{GR}}$ or R^∞). We start with a general lemma about tight closure that includes the hypothesis that R contains a test element. Recall from Hochster and Huneke's Theorem 2.2.6, if R is a reduced excellent local ring, R will always contain a test element.

Lemma 3.2.5. *Let* (R, m) *be a reduced local ring of positive characteristic* p *that has a test element. Let* M *be a finitely generated* R*-module. Then* $u \in 0_M^*$ *if and only if* $u \in \bigcap_k (m^k M)_M^*$.

Proof. Let c be a test element in R. Then $u \in 0_M^*$ if and only if $c^{1/q} \otimes u = 0$ in $R^{1/q} \otimes M$ for all q by Lemma 2.2.8. This holds if and only if $c^{1/q} \otimes u \in \bigcap_k m^k(R^{1/q} \otimes M)$ for all q since $(R^{1/q}, m^{1/q})$ is also local, $R^{1/q} \otimes M$ is a finitely generated $R^{1/q}$-module, and the powers of $mR^{1/q}$ are cofinal with the powers of $m^{1/q}$. Since $m^k(R^{1/q} \otimes M) = \mathrm{Im}(R^{1/q} \otimes m^k M \to R^{1/q} \otimes M)$, the above occurs if and only if

$$c^{1/q} \otimes u \in \mathrm{Im}(R^{1/q} \otimes m^k M \to R^{1/q} \otimes M),$$

for all k and all q. Finally, since c is a test element, the previous holds if and only if $u \in (m^k M)_M^*$ for all k. $\square$

We now show an equivalence between the m-adic closure of modules in certain ring extensions and the Frobenius, plus, and graded-plus closures.

Lemma 3.2.6. *Let* R *be a reduced ring,* I *an ideal, and* $S = R^\infty$ *(respectively,* R *is also a domain and* $S = R^+$ *or* R *is also a graded domain and* $S = R^{+\mathrm{GR}}$ *or*

$S = R^{+\mathrm{gr}}$). Then $u \in (I^kM)^F_M$ (resp., $u \in (I^kM)^+_M$ or $u \in (I^kM)^{+\mathrm{gr}}_M$) for all k if and only if $1 \otimes u \in \bigcap_k I^k(S \otimes M)$.

Proof. By definition, $u \in (I^kM)^F_M$ (resp., $u \in (I^kM)^+_M$ or $u \in (I^kM)^{+\mathrm{gr}}_M$) if and only if $1 \otimes u \in \mathrm{Im}(S \otimes I^kM \to S \otimes M)$. This holds if and only if $1 \otimes u \in I^k(S \otimes M)$. Therefore, $u \in (I^kM)^F_M$ (resp., $u \in (I^kM)^+_M$ or $u \in (I^kM)^{+\mathrm{gr}}_M$) for all k if and only if $1 \otimes u \in \bigcap_k I^k(S \otimes M)$. □

We now give the promised result connecting tight closure in R_m and the m-adic closure in R^+ (resp., $R^{+\mathrm{GR}}$, $R^{+\mathrm{gr}}$, or R^∞).

Proposition 3.2.7. *Let R be a reduced ring of characteristic $p > 0$. Let m be a maximal ideal of R such that R_m has a test element (e.g., this holds if R_m is excellent). Let $S = R^\infty$ (resp., let R also be a domain and $S = R^+$ or let R be a graded domain and $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$). Moreover, let R be such that Frobenius closure (resp., plus closure or graded-plus closure) equals tight closure for finitely generated modules with m-coprimary quotient. Then for any finitely generated $N \subseteq M$ and $u \in M$, we have $u/1 \in (N_m)^*_{M_m}$ if and only if $1 \otimes \overline{u}$ is in the m-adic closure of $S \otimes M/N$. For M free, we further note that $(N_m)^*_{M_m} \cap M = \bigcap_k (N + m^kM)S \cap M$.*

Proof. Since $x \in (N_m)^*_{M_m}$ if and only if $\overline{x} \in 0^*_{M_m/N_m}$ and $M_m/N_m \cong (M/N)_m$, it is enough to show this for the case $N = 0$. By Lemma 3.2.5, $u/1 \in 0^*_{M_m}$ if and only if $u/1 \in \bigcap_k (m^kM_m)^*_{M_m}$. Since M/m^kM is clearly m-coprimary, [HH1, Proposition 8.9] shows that the contraction of $(m^kM_m)^*_{M_m}$ to M is just $(m^kM)^*_M$ for all k. Hence $u/1 \in 0^*_{M_m}$ if and only if $u \in \bigcap_k (m^kM)^*_M$. By our hypothesis, this holds if and only if $u \in \bigcap_k (m^kM)^F_M$ (resp., $u \in \bigcap_k (m^kM)^+_M$ or $u \in \bigcap_k (m^kM)^{+\mathrm{gr}}_M$). Then Lemma 3.2.6 shows this is equivalent to $1 \otimes u \in \bigcap_k m^k(S \otimes M)$.

In the case that M is free, the above shows $u \in (N_m)^*_{M_m} \cap M$ if and only if

$1 \otimes \overline{u} \in \bigcap_k m^k(S \otimes M/N)$, but $m^k(S \otimes M/N) \cong m^k(MS/NS)$ in this case. Further, $\overline{u} \in m^k(MS/NS)$ if and only if $u \in (N + m^k M)S$. $\square$

3.3 Computing Injective Hulls

In this section we study the injective hull of the residue field of R^∞, R^+, and $R^{+\mathrm{GR}}$, where R has positive characteristic and is a complete local domain or a standard graded K-algebra domain. Recall that for any ring A and A-module M, the *injective hull* of M, denoted by $E_A(M)$, is a maximal essential extension of M and may be thought of as the smallest injective module that contains M, since it is a direct summand of any other injective module containing M.

We start by studying the injective hull, $E_{R^\infty}(K^\infty)$, where $R = K[[x_1, \ldots, x_n]]$ or $R = K[x_1, \ldots, x_n]$. In the case of $\dim(R) = 1$, M. McDermott computed the injective hull in a rather concrete manner involving formal sums such that the support of each sum is a well-ordered set. See [McD, Proposition 5.1.1]. For $\dim(R) \geq 2$, we conjecture that the injective hull can similarly be written as a set of formal sums with support that has DCC. We present the progress made so far and a condition that must be satisfied if the conjecture is false. The results in dimension $n \geq 2$, however, show that there are elements of $E_{R^\infty}(K^\infty)$ that are not killed by any power of the maximal ideal of R. This contrasts with the Noetherian case where no such elements exist. We then show that this result also holds for complete local domains and standard graded K-algebra domains, in positive characteristic, and that it holds for the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$ as well.

This latter result shows that we cannot extend Theorem 3.1.6 by making use of the injective hull of the residue field of $R^{+\mathrm{GR}}$ or R^∞ in the analogous way. The strategy of the proof of Theorem 3.1.6 was to show that for a module M with an element u outside of $0_M^{+\mathrm{gr}}$ or 0_M^F, we can map M to a m_R-coprimary graded module M' such that the image of u remains outside of the closure of 0 in M'. If the injective hull of the residue field in $R^{+\mathrm{GR}}$ or R^∞ was such that every element was killed by a power of m_R, then we could use a finitely generated R-submodule of $E_{R^\infty}(K^\infty)$ as M'. Propositions 3.3.12 and 3.3.13, however, bar us from applying this method to the problem.

We start with the case of $R = K[[x_1, \ldots, x_n]]$ or $R = K[x_1, \ldots, x_n]$ and look at essential extensions of K^∞ over R^∞.

3.3.1 The Regular Case

In order to study the injective hull $E_{R^\infty}(K^\infty)$, where $R = K[[x_1, \ldots, x_n]]$ or $R = K[x_1, \ldots, x_n]$, we will construct a module of formal sums such that the support has DCC. The supports will be subsets of $-\mathbb{N}[1/p]^n$, the set of n-tuples of nonpositive rational numbers whose denominators are powers of p. We are able to show that this module is an essential extension of K^∞, but we are unable to decide whether it is the entire injective hull or not in dimension $n \geq 2$.

Throughout the rest of this section, we will use bold letters to stand for n-tuples of elements. For instance, $\mathbf{x} := x_1, \ldots, x_n$, and $\mathbf{a} := a_1, \ldots, a_n$. We will place a partial ordering on n-tuples by comparing coordinate-wise, e.g., $\mathbf{a} > \mathbf{b}$ if and only if $a_i \geq b_i$, for all i, and $a_j > b_j$, for some j. We will define addition and subtraction of n-tuples as usual. If $\mathbf{a} \in \mathbb{Q}^n$, then $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$.

Definition 3.3.1. Let $R = K[[\mathbf{x}]]$ or $R = K[\mathbf{x}]$, where K is a field of positive characteristic p and $\dim R = n$. Let $L = K^\infty$, the perfect closure of K. Given a formal sum $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}}$, where $\mathbf{a} \in \mathbb{N}[1/p]^n$ and $c_{\mathbf{a}} \in L$, we will say that the *support* of f is the subset of $(-\mathbb{N}[1/p])^n$ given by

$$\operatorname{supp}(f) := \{-\mathbf{a} \,|\, c_{\mathbf{a}} \neq 0\}.$$

Using the same notation, we define the following set of formal sums

$$N := \{f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}} \,|\, \mathbf{a} \in \mathbb{N}[1/p],\, c_{\mathbf{a}} \in L, \text{ and} \operatorname{supp}(f) \text{ has DCC}\}.$$

Lemma 3.3.2. *Using the notation of Definition 3.3.1, N is an S-module with formally defined multiplication.*

Proof. Let $f_1,\ f_2$ be in N, and let $\operatorname{supp}(f_i) = A_i$. Then $\operatorname{supp}(f_1 + f_2) \subseteq A_1 \cup A_2$. Since the union of two sets with DCC has DCC and a subset of a set with DCC also has DCC, $f_1 + f_2$ is in N. Now, let $s \in S$. Then $s \in R^{1/q}$, for some $q = p^e$, so that we can write

$$s = \sum_{\mathbf{b} \geq 0} d_{\mathbf{b}} \mathbf{x}^{\mathbf{b}/q},$$

where $\mathbf{b} \in \mathbb{N}^n$ and $d_{\mathbf{b}} \in L$. Put

$$f := \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}} \in N.$$

Using formal multiplication, the coefficient of $\mathbf{x}^{-\mathbf{s}}$ in sf is

$$\sum_{-\mathbf{a}+\mathbf{b}/q=-\mathbf{s}} c_{\mathbf{a}} d_{\mathbf{b}}. \tag{3.3.3}$$

Notice that the coefficient of $\mathbf{x}^{-\mathbf{s}}$ is 0 if $-\mathbf{s} = -\mathbf{a} + \mathbf{b}/q > 0$ as $K^{\infty} = S/m_S$.

When $-\mathbf{s} \leq 0$, for sf to be well-defined, the summation (3.3.3) must consist of a finite sum of nonzero elements. In the polynomial case, this is clear since s has only finitely many terms. Otherwise, suppose that we have enumerated the terms contributing to the coefficient of $\mathbf{x}^{-\mathbf{s}}$ and that the set

$$\{k \in \mathbb{N} \,|\, -\mathbf{a}^{(k)} + \mathbf{b}^{(k)}/q = -\mathbf{s},\ \text{and } c_{\mathbf{a}} d_{\mathbf{b}} \neq 0\}$$

is infinite. If there are only finitely many distinct $\mathbf{b}^{(k)}$, then (3.3.3) is clearly a finite sum. We may then assume that there are infinitely many distinct $\mathbf{b}^{(k)}$ and thus assume that all of the $\mathbf{b}^{(k)}/q$ are distinct. Hence, we obtain an infinite chain of equalities

$$-\mathbf{a}^{(1)} + \mathbf{b}^{(1)}/q = -\mathbf{a}^{(2)} + \mathbf{b}^{(2)}/q = -\mathbf{a}^{(3)} + \mathbf{b}^{(3)}/q = \cdots .$$

Since the sets $\mathbb{N}/q$ and $\mathrm{supp}(f)$ have DCC, we may apply Lemma 3.3.4, and then we will have a contradiction. Therefore, (3.3.3) is a finite sum, and sf is well-defined.

We also need to show that $\mathrm{supp}(sf)$ has DCC. Suppose to the contrary that

$$-\mathbf{a}^{(1)} + \mathbf{b}^{(1)}/q > -\mathbf{a}^{(2)} + \mathbf{b}^{(2)}/q > -\mathbf{a}^{(3)} + \mathbf{b}^{(3)}/q > \cdots .$$

is an infinite chain in $\mathrm{supp}(sf)$. If there are only finitely many distinct n-tuples $\mathbf{b}^{(k)}/q$, then we also obtain an infinite descending chain in the $-\mathbf{a}^{(k)}$, for $k \gg 0$, a contradiction since $\mathrm{supp}(f)$ has DCC. We may then assume that there are infinitely many $\mathbf{b}^{(k)}/q$ and all are distinct and then apply Lemma 3.3.4 again to obtain a contradiction. $\square$

Lemma 3.3.4. *Let A and B be subsets of G^n, where $(G, +)$ is a linearly ordered abelian group. Suppose that A has DCC and that B has DCC in each coordinate. If $\{\mathbf{a}^{(k)}\}_k$ is a sequence of n-tuples in A and $\{\mathbf{b}^{(k)}\}_k$ is a sequence of infinitely many distinct n-tuples in B, then we cannot obtain an infinite chain*

$$\mathbf{a}^{(1)} + \mathbf{b}^{(1)} \geq \mathbf{a}^{(2)} + \mathbf{b}^{(2)} \geq \mathbf{a}^{(3)} + \mathbf{b}^{(3)} \geq \cdots .$$

Proof. Because each $\mathbf{b}^{(k)}$ has only finitely many coordinates and each $\mathbf{b}^{(k)}$ is distinct, $b_i^{(k)}$ takes on infinitely many values, for some i. Without loss of generality, we may assume that $b_1^{(k)}$ is distinct, for all k, by taking subsequences. Similarly, if any other coordinate $b_j^{(k)}$ takes on infinitely many values, then we may also assume that each

$b_j^{(k)}$ is distinct, for all k. If any coordinate $b_j^{(k)}$ takes on only finitely many values, then it must take on a particular value infinitely many times. So, by taking subsequences again, we may assume that $b_j^{(k)}$ is constant, for all k.

Therefore, after taking subsequences, we may assume without loss of generality that, for each i, either $b_i^{(k)} = b_i^{(k')}$, or $b_i^{(k)} < b_i^{(k+1)}$, for all k, k'. (The latter assumption may be made when there are infinitely many distinct values because B has DCC in each coordinate.)

These conditions imply that we obtain a chain $\mathbf{b}^{(1)} < \mathbf{b}^{(2)} < \mathbf{b}^{(3)} < \cdots$, and so if we subtract this line of inequalities from our original chain of inequalities, we obtain an infinite descending chain $\mathbf{a}^{(1)} > \mathbf{a}^{(2)} > \mathbf{a}^{(3)} > \cdots$, which contradicts the fact that A has DCC. Therefore, we could not have had the original infinite chain. $\square$

Now that we have established that N is actually an R^∞-module, we can also show that N is an essential extension of K^∞.

Proposition 3.3.5. *Using the notation of Definition 3.3.1, N is an essential extension of $L = K^\infty$. Therefore, $N \subseteq E_{R^\infty}(K^\infty)$.*

Proof. The second claim follows immediately from the first. For the first, let $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}} \in N$. Since $\operatorname{supp}(f)$ has DCC, we can choose a minimal element $-\mathbf{a}^{(0)}$. Then $\mathbf{x}^{\mathbf{a}^{(0)}} \in S$, and

$$\mathbf{x}^{\mathbf{a}^{(0)}} f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}^{(0)} - \mathbf{a}} = c_{\mathbf{a}^{(0)}} \in L \setminus \{0\}$$

as $\mathbf{a}_i^{(0)} > \mathbf{a}_i$, for some i, for all $\mathbf{a} \neq \mathbf{a}^{(0)}$ in $\operatorname{supp}(f)$. $\square$

As mentioned earlier, McDermott showed that N is the entire injective hull of K^∞ over R^∞ in dimension 1. McDermott's proof covers that case $R = K[x]$, but the case $R = K[[x]]$ follows routinely.

Proposition 3.3.6 (Proposition 5.1.1, [McD])**.** *With the notation of Definition 3.3.1, the injective hull $E_S(L)$ is isomorphic to N when* $\dim R = n = 1$.

Proposition 3.3.5 has the following corollary which demonstrates that in the non-Noetherian case the injective hull of a residue field behaves very differently from the Noetherian case in dimension $n \geq 2$, where all elements of the injective hull are killed by a power of the maximal ideal.

Proposition 3.3.7. *With the notation of Definition 3.3.1, if $n \geq 2$, then $E_{R^\infty}(K^\infty)$ contains an element not killed by any power of $m_R = (\mathbf{x})R$.*

Proof. Let $f = \sum_e x_1^{-1/p^e} x_2^{-e}$. For $e < e'$, $-1/p^e < -1/p^{e'}$ and $-e > -e'$ so that all elements in $\operatorname{supp}(f)$ are incomparable. Hence, all chains in $\operatorname{supp}(f)$ have only one link, and $f \in N$, which injects into $E_{R^\infty}(K^\infty)$ by the last proposition. Now, let $t > 0$. Then $x_2^t f = \sum_e x_1^{-1/p^e} x_2^{n-e}$, and if $e_0 \geq t$, then $t - e_0 \leq 0$. Therefore, $x_2^t f \neq 0$, and $m_R^t f \neq 0$, for any $t > 0$. □

We now prove some additional facts about the module N that may be useful in showing it is the full injective hull of $L = K^\infty$. We first show that N embeds in the injective $S = R^\infty$-module $\operatorname{Hom}_L(S, L)$ because it is an essential extension of L. This fact implies that in order to prove that N is isomorphic to $E_S(L)$, it is enough to show that N has no essential extension in $\operatorname{Hom}_L(S, L)$.

Lemma 3.3.8. *With the notation of Definition 3.3.1, there exists an injective map $\theta : N \hookrightarrow \operatorname{Hom}_L(S, L)$ that takes $1 \in L$ inside N to the map $S \to L$ taking $1 \mapsto 1$ and $m_S \mapsto 0$, where $m_S = \bigcup_q (\mathbf{x}^{1/q})S$.*

Proof. Let $\lambda : N \to L$ be the L-linear map given by $\lambda(\sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}}) = c_{\mathbf{0}}$, and let $\theta'' : N \times S \to L$ be given by $\theta''(f, s) = \lambda(sf)$. It is easy to check that θ'' is an

L-bilinear map and so induces the map $\theta' : N \otimes_L S \to L$ given by $\theta'(f \otimes s) = \lambda(sf)$. By the adjointness of tensor and Hom, we have

$$\mathrm{Hom}_L(N \otimes_L S, L) \cong \mathrm{Hom}_L(N, \mathrm{Hom}_L(S, L)).$$

Therefore, θ' corresponds to a unique L-linear map $\theta : N \to \mathrm{Hom}_L(S, L)$.

We claim that θ is actually S-linear. Indeed, using the adjointness, $\theta(f)(s) = \lambda(sf)$. Let $r \in S$. Using the inherited S-module structure on $\mathrm{Hom}_L(S, L)$, we have

$$r\theta(f)(s) = \theta(f)(sr) = \lambda((sr)f) = \lambda(s(rf)) = \theta(rf)(s).$$

Therefore, $r\theta(f) = \theta(rf)$ and θ is S-linear.

Now, $\theta(1)(1) = \lambda(1) = 1 \neq 0$, and $\theta(1)(m_S) = \lambda(m_S) = 0$. This shows $\theta(1) \neq 0$ so that $\theta|_L : L \hookrightarrow \mathrm{Hom}_L(S, L)$ according to the second assertion above. Additionally, since N is an essential extension of L, we have $\theta : N \hookrightarrow \mathrm{Hom}_L(S, L)$. $\square$

We next show that if N has an essential extension within the set of formal sums, then there must be such a sum that has no minimal elements in its support. First we need a lemma about sets with DCC.

Lemma 3.3.9. *Let A be a partially ordered set such that for all $B \subsetneq A$, where B has DCC, the difference $A \setminus B$ has a minimal element. Then A has DCC.*

Proof. Let $C := \{a \in A \mid \text{there exists an infinite chain } a > a_1 > a_2 > \cdots \text{ in } A\}$. Suppose that A does not have DCC. Then C is not empty. Moreover, C does not have a minimal element because all of its members begin an infinite chain. If we set $B = A \setminus C$, then by our hypothesis, B cannot have DCC. Thus, there exists an infinite chain $b_0 > b_1 > b_2 > \cdots$ in $B \subsetneq A$, and so $b_0 \in C \cap B = \varnothing$, a contradiction. $\square$

Lemma 3.3.10. *Using the notation of Definition 3.3.1, let $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}}$, where*

$\mathrm{supp}(f) \subseteq (-\mathbb{N}[1/p])^n$. *If the support of* $sf + g$ *has a minimal element, for all* $s \in S$ *and all* $g \in N$*, then* $f \in N$.

Proof. Let $B \subsetneq \mathrm{supp}(f)$ such that B has DCC. Then $g = \sum_{-\mathbf{a} \in B} -c_{\mathbf{a}} \mathbf{x}^{-\mathbf{a}}$ is in N. Therefore, by hypothesis, $\mathrm{supp}(f+g)$ has a minimal element, but $\mathrm{supp}(f+g) = \mathrm{supp}(f) \setminus B$. By the previous lemma, $\mathrm{supp}(f)$ has DCC, and so $f \in N$. □

3.3.2 The General Case

We will now show how we can extend the result of Proposition 3.3.7 to include complete local domains and standard graded K-algebra domains in positive characteristic. Moreover, we will also extend the result to one concerning the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$. These results will then show that in a large class of rings, there are elements not killed by any power of the maximal ideal of R in the injective hull of the residue field over R^∞, R^+, or $R^{+\mathrm{GR}}$.

We will first need a lemma about purity. An injection of R-modules $N \to M$ is called *pure* if $W \otimes N \to W \otimes M$ is an injection for all R-modules W. When M/N is finitely presented, the map is pure if and only if the map splits; see [HR, Corollary 5.2]. When S is an R-algebra and $R \to S$ is pure as a map of R-modules, one calls S *pure* over R.

Lemma 3.3.11. *Let* $R = \varinjlim_\alpha R_\alpha$*, and let* $S = \varinjlim_\alpha S_\alpha$ *such that each* S_α *is pure over* R_α*. Then* S *is pure over* R.

Proof. Since each map $R_\alpha \to S_\alpha$ is injective by hypothesis, it is clear that $R \to S$ is also injective.

Let W be an R-module and thus an R_α-module, for each α, by restriction of scalars. Suppose that $W \to W \otimes_R S$ is not an injective map. Then there exists $w \in W$ such that $w \otimes 1 = 0$ in $W \otimes_R S$. It is easy to check that $W \otimes_R S = \varinjlim_\alpha W \otimes_{R_\alpha} S_\alpha$, and so $w \otimes 1 = 0$ in some $W \otimes_{R_\alpha} S_\alpha$. Hence, as a map of R_α-modules, $W \to W \otimes_{R_\alpha} S_\alpha$ is not injective, a contradiction. $\square$

Therefore, if A is a regular ring of positive characteristic and R is a reduced module-finite extension of A, then A is a direct summand of R as an A-module (see [Ho1, Theorem 1]). Thus, $A^{1/q}$ is a direct summand of $R^{1/q}$, for all $q = p^e$, and so the last lemma implies that R^∞ is pure over A^∞.

Proposition 3.3.12. *Let (R, m, K) be a complete local domain (resp., a standard graded K-algebra domain) of positive characteristic and Krull dimension $n \geq 2$. Then there exists an element of $E := E_{R^\infty}(K^\infty)$ that is not killed by any power of m.*

Proof. By the Cohen structure theorem, R is a module-finite extension of a formal power series ring $A = K[[x_1, \ldots, x_n]]$ (resp., by Noether normalization, R is a module-finite extension of the graded polynomial ring $A = K[x_1, \ldots, x_n]$). Since A is regular, R is pure over A, and so the last lemma implies that R^∞ is pure over A^∞. If we let $E_0 := E_{A^\infty}(K^\infty)$, then

$$K^\infty \hookrightarrow E_0 \hookrightarrow M := R^\infty \otimes_{A^\infty} E_0.$$

Since K^∞ is an R^∞-module, we can find an R^∞-submodule M' of M maximal with respect to not intersecting K^∞. Hence, M/M' is an essential extension of K^∞ as an R^∞-module. We can then extend M/M' to a maximal essential extension E of K^∞ over R^∞. Since the inclusion $K^\infty \to E$ factors through E_0 and since E_0 is an essential extension of K^∞ over A^∞, E_0 injects into E as a map of A^∞-modules. Since E_0 contains an element not killed by any power of the maximal ideal (resp., the homogeneous maximal ideal) m_A of A by Proposition 3.3.7, so does E. Since

m_A is primary to m, the same element of E not killed by a power of m_A is also not killed by a power of m. $\square$

We can also take advantage of the faithful flatness of A^+ or $A^{+\mathrm{GR}}$ over a regular ring A (see Proposition 2.3.2) to prove the existence of elements not killed by a power of the maximal ideal in the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$.

Proposition 3.3.13. *Let (R, m) be a complete local domain (resp., a standard graded K-algebra domain) of positive characteristic and Krull dimension $n \geq 2$. Then there exists an element of $E := E_{R^+}(\overline{K})$ (resp., $E := E_{R^{+\mathrm{GR}}}(\overline{K})$) that is not killed by any power of m, where $\overline{K}$ is the algebraic closure of K.*

Proof. By the Cohen structure theorem, R is a module-finite extension of a formal power series ring $A = K[[x_1, \ldots, x_n]]$ (resp., by Noether normalization, R is a module-finite extension of the graded polynomial ring $A = K[x_1, \ldots, x_n]$). Thus, $A^+ \cong R^+$ (resp., $A^{+\mathrm{GR}} \cong R^{+\mathrm{GR}}$), and so we may assume that $R = K[[x_1, \ldots, x_n]]$ (resp., $R = K[x_1, \ldots, x_n]$). Let $B := R^+$ (resp., $B := R^{+\mathrm{GR}}$).

Since $R^{1/q}$ is regular, for all q, and since B is a big Cohen-Macaulay $R^{1/q}$-algebra, B is faithfully flat over $R^{1/q}$ by Proposition 2.3.2. Therefore, B is flat over R^∞ (using a simple direct limit argument).

The inclusion of $K^\infty \subseteq E_{R^\infty}(K^\infty)$, together with the flatness of B over R^∞ gives the following diagram:

$$\begin{tikzcd} K^\infty \arrow[r, hook] \arrow[d] & E_{R^\infty}(K^\infty) \arrow[d] \\ B \otimes_{R^\infty} K^\infty \arrow[r, hook] & B \otimes_{R^\infty} E_{R^\infty}(K^\infty) \end{tikzcd}$$

As we have a surjection of $B \otimes_{R^\infty} K^\infty$ onto $\overline{K}$, the residue field of B, we have a map from $B \otimes_{R^\infty} K^\infty$ to E, the injective hull of $\overline{K}$ over B. Because E is injective, this map lifts to a map from $B \otimes_{R^\infty} E_{R^\infty}(K^\infty)$. Hence, we obtain a commutative diagram of R^∞-module maps:

$$\begin{tikzcd} K^\infty \arrow[r, hook] \arrow[d, hook] & E_{R^\infty}(K^\infty) \arrow[dl] \\ E & \end{tikzcd}$$

where the diagonal map is also injective since $E_{R^\infty}(K^\infty)$ is an essential extension of K^∞. Therefore, the element in $E_{R^\infty}(K^\infty)$ not killed by any power of m (as in Proposition 3.3.7) injects into E as an element not killed by any power of m. $\square$

CHAPTER 4

Graded-Complete Big Cohen-Macaulay Algebras

Our goal in this chapter is to create a balanced big Cohen-Macaulay algebra B for an $\mathbb{N}$-graded ring R with homogeneous maximal ideal m such that if M is a free R-module and $N \subseteq M$, then $NB \cap M = (N_m)^*_{M_m} \cap M$. In other words, extension and contraction with respect to B will characterize tight closure in R_m. We will start by defining the concept of a *graded-complete* ring which is an analogue, for graded rings, of complete local rings. This will be used to define a class of *lim-graded-complete* rings which are direct limits of graded-complete rings. We then construct lim-graded-complete R-algebras that are also big Cohen-Macaulay algebras for R. Finally we show that these algebras have the property claimed above. The methods used will closely follow those used in [HH7, Sections (3.1)-(3.7)] and in the proof of [Ho3, Theorem 11.1].

The motivation behind the study of graded-complete rings and modules comes from our attempts to extend Theorem 3.1.6 to modules that are not necessarily m-coprimary. We had hoped that the construction of a particular graded-complete module would allow us to extend our result. So far these attempts have been unsuccessful.

4.1 Graded-Complete and Lim-Graded-Complete Rings

In this section we will define a completion operation for graded rings that is analogous to the completion operation for maximal ideals of Noetherian rings. In fact, in the case that R_0 is a field and R is Noetherian, these operations coincide. We then use this operation to define a class of *graded-complete* rings that have certain properties of graded rings, but also share some properties with complete rings. We will also discuss direct limits of graded-complete rings which we will call *lim-graded-complete* and how graded-completions and direct limits can preserve the property of being a big Cohen-Macaulay algebra. Finally, we will introduce *graded-complete modules.*

In the following sections we will mostly work with $\mathbb{N}$-graded rings and occasionally with $\mathbb{N}\frac{1}{s}$ or $\mathbb{Z}\frac{1}{s}$-graded rings, where $\mathbb{Z}\frac{1}{s} := \{\frac{k}{s} | k \in \mathbb{Z}\}$, $\mathbb{N}\frac{1}{s} := \{\frac{n}{s} | n \in \mathbb{N}\}$, when s is some positive integer.

Definition 4.1.1. If $R = \bigoplus_{i \in \mathbb{Z}\frac{1}{s}} R_i$ is a $\mathbb{Z}\frac{1}{s}$-graded (not necessarily Noetherian) ring, then we call $\widehat{R} := (\bigoplus_{i<0} R_i) \oplus (\prod_{i \geq 0} R_i)$ the *graded-completion* of R. A ring S is called $\mathbb{Z}\frac{1}{s}$*-graded-complete* if $S = \widehat{R}$ for a $\mathbb{Z}\frac{1}{s}$-graded ring R.

We can identify $\widehat{R}$ with the subgroup of $\prod_{i \in \mathbb{Z}\frac{1}{s}} R_i$ of elements with i^{th} coordinate $r_i = 0$ for all $i \ll 0$. We will write $\sum_i r_i$ for an element of $\widehat{R}$ in order to emphasize the analogy with the graded case. Here it is understood that $r_i \in R_i$, and $r_i = 0$ for all $i \ll 0$. It is clear that $\widehat{R}$ is an abelian group with addition performed componentwise, but we can also define a multiplication operation for $\widehat{R}$.

Lemma 4.1.2. *If R is a $\mathbb{Z}\frac{1}{s}$-graded ring, then $\widehat{R}$ is a commutative ring with identity, where multiplication is defined by $(\sum_i r_i)(\sum_j s_j) = (\sum_h t_h)$ such that $t_h = \sum_{\{i+j=h\}} r_i s_j$.*

The proof is clear, noting that because $r_i = 0$ and $s_j = 0$ for $i, j \ll 0$, the sum defining t_h is finite and so well-defined.

To justify the notation $\widehat{R}$, we note the following:

Lemma 4.1.3. *If R is a Noetherian $\mathbb{N}$-graded K-algebra such that $R_0 = K$, then $\prod_{i\in\mathbb{N}} R_i \cong \varprojlim_i R/m_i \cong (R_m)\widehat{\ }$, where $m = \bigoplus_{i\geq 1} R_i$ and $m_i = \bigoplus_{j\geq i} R_j$. Therefore, in such cases, $\widehat{R}$, as defined above, is a complete local ring.*

Proof. Since $R_0 = K$, m is the unique homogeneous maximal ideal of R. Then the second isomorphism follows because $R_m/m_iR_m \cong R/m_i$. For the first isomorphism, note that $\varprojlim_i R/m_i = \varprojlim_i R_0 \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_{i-1} \cong \prod_i R_i$. The final claim follows since $(R_m)\widehat{\ }$ is a complete local ring. $\square$

The following result will allow us to define a notion of degree for certain elements of $\widehat{R}$ and gives a functoriality result for the graded-completion operation.

Lemma 4.1.4. *Let R and S be $\mathbb{Z}\frac{1}{s}$-graded rings.*

(a) *The map $\psi_1 : R \to \widehat{R}$ given by $\psi_1(r) = \sum_i r_i$, where r_i is the degree i piece of r, is an injective R-algebra homomorphism.*

(b) *Given a degree-preserving map $\phi : R \to S$, ϕ extends to a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$ and gives a commutative diagram such that R_i in R maps into S_i in $\widehat{S}$ for all i.*

$$\begin{tikzcd} \widehat{R} \arrow[r, "\widehat{\phi}"] & \widehat{S} \\ R \arrow[u, "\psi_1"] \arrow[r, "\phi"] & S \arrow[u, "\psi_2"'] \end{tikzcd}$$

(c) *If ϕ, as in (b), is injective (resp., surjective), then so is $\widehat{\phi}$.*

Proof. The existence and injectivity in (a) of the map ψ is clear. For (b), define $\widehat{\phi}(\sum_i r_i) = \sum_i \phi(r_i)$, where $\deg r_i = i$. Since all maps clearly commute for homogeneous elements of R, the diagram commutes in general. Since ϕ is a degree-preserving map, ψ_1 maps R_i in R to R_i in $\widehat{R}$, and similarly for ψ_2 and S, we see that the composition $R \to \widehat{S}$ does map R_i into S_i for all i. Finally, since $\widehat{\phi}(\sum_i r_i)$ is determined by the images $\widehat{\phi}(r_i) = \phi(r_i)$ in S_i, the claims in (c) follow. $\square$

We now define the notion of degree for $\widehat{R}$ and the notion of a map that preserves degree.

Definition 4.1.5. Given $\sum_i r_i \in \widehat{R}$, we call r_i the *degree i homogeneous piece* (or *component*) of $\sum_i r_i$. We will call an element $\sum_i r_i$ *homogeneous* of degree i if $r_i \neq 0$ and $r_j = 0$ for all $j \neq i$. A homomorphism $\phi : R \to \widehat{S}$ or $\phi : \widehat{R} \to \widehat{S}$ is called *degree-preserving* if R_i maps into S_i for all i and $\phi(\sum_i r_i) = \sum_i \phi(r_i)$.

From the definition, it is clear that in $\widehat{R}$ we have $R_i R_j \subseteq R_{i+j}$, as this is the case in R, and that all elements of $\widehat{R}$ are (possibly infinite) sums of homogeneous elements with only finitely many nonzero components in negative degrees. We also note that the natural map $\phi : R \to \widehat{R}$ is degree-preserving, and if $R \to S$ is a degree-preserving map of graded rings, then the induced maps $R \to \widehat{S}$ and $\widehat{R} \to \widehat{S}$ are degree-preserving.

It is because of the notion of degree defined above and the result of Lemma 4.1.3 that we call the rings $\widehat{R}$ graded-complete. Notice that Lemma 4.1.4 shows that the graded-completion operation is a functor, and using our definition of degree-preserving maps, we see that the category of graded rings with degree-preserving maps is actually equivalent to the category of graded-complete rings with degree-preserving maps. Since any $\mathbb{N}\frac{1}{s}$-graded ring is naturally $\mathbb{Z}\frac{1}{s}$-graded as well, and, if s' divides s, then a $\mathbb{Z}\frac{1}{s'}$-graded ring is naturally $\mathbb{Z}\frac{1}{s}$-graded, we can say the same thing about $\mathbb{N}\frac{1}{s}$ and $\mathbb{Z}\frac{1}{s}$-graded-complete rings.

It will be useful in the following sections to work also with direct limits of graded-complete rings so that we can define degree and, thus, degree-preserving maps for such direct limits. Let $\{\widehat{R}_\alpha\}_\alpha$ be a directed system of $\mathbb{Z}\frac{1}{s}$-graded-complete rings such that all maps $\widehat{R}_\alpha \to \widehat{R}_\beta$ are degree-preserving. Let $S = \varinjlim_\alpha \widehat{R}_\alpha$ in the category of rings. Given an element $s \in S$, choose a representative $\sum_i r_i^{(\alpha)}$ in $\widehat{R}_\alpha$. We will call the image of $r_i^{(\alpha)}$ in S the *degree i homogeneous piece* of s. Because all maps in the direct limit system are degree-preserving, it is clear that the degree i homogeneous piece of s is independent of the choice of representative of s. If we let S_i be the abelian group generated by all homogeneous elements of S of degree i, for all $i \in \mathbb{Z}\frac{1}{s}$, then we also have $S_i S_j \subseteq S_{i+j}$ since this is true in each $\widehat{R}_\alpha$. Further, since each element of each $\widehat{R}_\alpha$ can be written as a (possibly infinite) sum of homogeneous elements, the same is true for S. Therefore, each element of S can be written as a possibly infinite sum $\sum_i s_i$ such that each s_i is the image of $r_i^{(\alpha)}$, for the same α, where $\sum_i r_i^{(\alpha)}$ is an element of $\widehat{R}_\alpha$.

Definition 4.1.6. A ring $S = \varinjlim_\alpha \widehat{R}_\alpha$, where each $\widehat{R}_\alpha$ is $\mathbb{Z}\frac{1}{s}$-graded-complete, will be called $\mathbb{Z}\frac{1}{s}$*-lim-graded-complete.*

Note that all graded-complete rings $\widehat{R}$ are also lim-graded-complete as $\widehat{R}$ is the direct limit of the directed system containing only $\widehat{R}$. We can now generalize our definition of a degree-preserving map to include any map to and from a graded ring or a lim-graded-complete ring.

Definition 4.1.7. Let R and S each be either graded or lim-graded-complete. A *degree-preserving* map is any homomorphism $\phi : R \to S$ such that R_i maps to S_i for all i, and $\phi(\sum_i r_i) = \sum_i \phi(r_i)$.

Remark 4.1.8. Direct limit and graded-completion do not commute although there is a natural map. Indeed, let $K^{(0)} \subset K^{(1)} \subset K^{(2)} \subset \cdots$ be a strictly increasing tower of fields. We then have the strict inclusion maps

$$(*) \qquad K^{(0)}[x] \hookrightarrow K^{(1)}[x] \hookrightarrow K^{(2)}[x] \hookrightarrow \cdots$$

of polynomial rings. Taking graded-completions we obtain the sequence of formal power series rings $K^{(0)}[[x]] \hookrightarrow K^{(1)}[[x]] \hookrightarrow \cdots$, and taking a direct limit produces the ring $R = \bigcup_i (K^{(i)}[[x]])$. If $L = \bigcup_i K^{(i)}$, then taking the direct limit of $(*)$ yields the polynomial ring $L[x]$ and taking the graded-completion gives the formal power series ring $L[[x]]$. Now, let a_0 be any element of $K^{(0)}$, and for $i > 0$, let a_i be an element of $K^{(i)} \setminus K^{(i-1)}$. Then $\sum_i a_i x^i$ is an element of $L[[x]]$ as each $a_i \in L$, but it is not an element of R since no $K^{(i)}$ contains every coefficient a_i.

Lemma 4.1.9. *Let $\{R_\alpha\}_\alpha$ be directed system of $\mathbb{Z}\frac{1}{s}$-graded rings such that all maps are degree-preserving. Then there exists a degree-preserving R-algebra map*

$$\Psi : \varinjlim_\alpha \widehat{R}_\alpha \longrightarrow (\varinjlim_\alpha R_\alpha)\widehat{}.$$

Proof. Let $s \in \varinjlim_\alpha \widehat{R}_\alpha$ be represented by $r = \sum_i r_i$ such that $r \in \widehat{R}_\alpha$ and $\deg r_i = i$. For each i, we have $r_i \in [R_\alpha]_i$, and so the image $\overline{r_i}$ is in $[\varinjlim_\alpha R_\alpha]_i$ for all i. Define $\Psi(s) = \sum_i \overline{r_i}$. The commutativity of the diagram in Lemma 4.1.4(b) implies that this map is independent of the choice of representative for s. □

It will also be useful to note that if R is Noetherian (respectively, a domain), then the graded-completion operation preserves this property.

Lemma 4.1.10. *If R is a Noetherian $\mathbb{Z}\frac{1}{s}$-graded ring, then $\widehat{R}$ is Noetherian.*

Proof. Without loss of generality, we can assume that R is $\mathbb{Z}$-graded. Since R is Noetherian, R is finitely generated over R_0 by [BH, Theorem 1.5.5], and so there exists a degree-preserving surjection $R_0[T_1, \ldots, T_n] \twoheadrightarrow R$, where the T_i are indeterminates over R_0. By Lemma 4.1.4(c),

$$R_0[[T_1, \ldots, T_n]] \cong (R_0[T_1, \ldots, T_n])\widehat{\ } \twoheadrightarrow \widehat{R}$$

so that $\widehat{R}$ is also Noetherian. $\square$

Lemma 4.1.11. *If R is a $\mathbb{Z}\frac{1}{s}$-graded domain, then $\widehat{R}$ is a graded-complete domain.*

Proof. If $\sum_i r_i$ and $\sum_j s_j$ are non-zero elements in $\widehat{R}$, let i_0 be the smallest non-negative integer i such that $r_i \neq 0$ and similarly define j_0 with respect to $\sum_j s_j$. Then the degree $i_0 + j_0$ entry of $(\sum_i r_i)(\sum_j s_j)$ is $r_{i_0} s_{j_0} \neq 0$ as R is a domain. Therefore, $(\sum_i r_i)(\sum_j s_j) \neq 0$ in $\widehat{R}$, which is then also a domain. $\square$

We now take a brief look at lim-graded-complete rings and big Cohen-Macaulay algebras over a graded ring R.

Definition 4.1.12. A *(possibly improper) $\mathbb{Z}\frac{1}{s}$-graded big Cohen-Macaulay R-algebra* is a $\mathbb{Z}\frac{1}{s}$-graded or lim-graded-complete degree-preserving R-algebra B such that every homogeneous system of parameters in R is a (possibly improper) regular sequence on B.

If S is any graded or lim-graded-complete R-algebra and $x_1, \ldots, x_k$ is part of a homogeneous system of parameters in R, then a relation $\sum_{j=1}^{k} x_j u_j = 0$ in S is called *trivial* if $u_k \in (x_1, \ldots, x_{k-1})S$.

Remark 4.1.13. Our definition implies that our graded big Cohen-Macaulay algebras are *balanced*, i.e., every homogeneous system of parameters in R is a regular sequence. Recall that some authors will call an R-algebra B a big Cohen-Macaulay algebra if a single (homogeneous) system of parameters is a regular sequence on B.

If every relation $\sum_{j=1}^{k} x_j u_j = 0$ in S for all homogeneous systems of parameters $x_1, \ldots, x_k$ in R is trivial, then S is a possibly improper graded big Cohen-Macaulay R-algebra.

We note two useful facts concerning graded big Cohen-Macaulay algebras and the graded-completion operation:

Lemma 4.1.14. *If R and S are $\mathbb{Z}\frac{1}{s}$-graded and S is a (possibly improper) graded big Cohen-Macaulay R-algebra, then $\widehat{S}$ is also.*

Proof. Since $\widehat{S}$ has a notion of degree, we can assume we have a homogeneous relation on a homogeneous system of parameters from R, which is a relation in S since it is composed of homogeneous pieces. Since the relation trivializes in S, it is trivial in $\widehat{S}$ as well.

If there exists a homogeneous system of parameters $x_1, \ldots, x_n$ in R such that $\widehat{S}/(x_1, \ldots, x_n)\widehat{S} = 0$, then there is a homogeneous relation $1 = \sum_j x_j s_j$ in $\widehat{S}$ which also holds in S since S injects into $\widehat{S}$ by Lemma 4.1.4(a). Thus, if S is a proper big Cohen-Macaulay algebra, then $\widehat{S}$ is also proper. $\square$

Lemma 4.1.15. *If R is $\mathbb{Z}\frac{1}{s}$-graded and S_α is a degree-preserving directed system of $\mathbb{Z}\frac{1}{s}$-graded R-algebras such that $\varinjlim_\alpha S_\alpha$ is a (possibly improper) graded big Cohen-Macaulay R-algebra, then $T = \varinjlim_\alpha \widehat{S}_\alpha$ is as well.*

Proof. Given a relation $\sum_{j=1}^k x_j u_j = 0$ on a homogeneous system of parameters $x_1, \ldots, x_k$ in R, we can assume that the relation is homogeneous since T is lim-graded-complete. Now, because T is a direct limit, there exists some $\widehat{S}_\alpha$ such that all u_j are in $\widehat{S}_\alpha$ and so that the relation $\sum_{j=1}^k x_j u_j = 0$ also holds. As in Lemma 4.1.4(a), the natural map $S_\alpha \to \widehat{S}_\alpha$ is injective and so the u_j are in S_α and $\sum_{j=1}^k x_j u_j = 0$ holds as well. Since $\varinjlim_\alpha S_\alpha$ is a (possibly improper) graded big Cohen-Macaulay R-algebra, this relation is trivial in some S_β and therefore in $\widehat{S}_\beta$ which means it is trivial in T.

If there exists a homogeneous system of parameters $x_1, \ldots, x_n$ in R such that

$T/(x_1, \ldots, x_n)T = 0$, then there is a homogeneous relation $1 = \sum_j x_j u_j$ in T which also holds in some $\widehat{S}_\alpha$. Thus, it holds in S_α since $S_\alpha \hookrightarrow \widehat{S}_\alpha$ and so holds in $\varinjlim_\alpha S_\alpha$. Thus, if $\varinjlim_\alpha S_\alpha$ is proper, then T is also proper. $\square$

4.2 Graded-Complete Modules

Although we will not make use of the following in the later sections, for a sense of completeness, we also define *graded-complete R-modules*.

Definition 4.2.1. If M is a $\mathbb{Z}\frac{1}{s}$-graded module over the $\mathbb{Z}\frac{1}{s}$-graded ring R, then we will call $\widehat{M} := (\bigoplus_{i<0} M_i) \bigoplus (\prod_{i \geq 0} M_i)$ the *graded-completion* of M.

We present the following list of properties without proof as the proofs are either straightforward or are direct analogues of properties for the graded-completion of a ring.

Lemma 4.2.2. *Let M, N, and Q be graded modules over a $\mathbb{Z}\frac{1}{s}$-graded ring R.*

(a) *$\widehat{M}$ is an $\widehat{R}$-module with an action induced by the R-module structure of M.*

(b) *There exists a natural map $\widehat{R} \otimes M \to \widehat{M}$.*

(c) *If $M \to N$ is a degree-preserving map of modules, then we have an induced commutative diagram*

$$\begin{array}{ccc} \widehat{M} & \longrightarrow & \widehat{N} \\ \uparrow & & \uparrow \\ \widehat{R} \otimes M & \longrightarrow & \widehat{R} \otimes N \end{array}$$

(d) *If we have degree-preserving maps $M \xrightarrow{g} N \xrightarrow{f} Q$, then $(f \circ g)\widehat{\ } = \widehat{f} \circ \widehat{g}$.*

(e) *The graded-completion operation preserves injections and surjections.*

(f) *If*

$$(\#) \qquad 0 \to M \to N \to Q \to 0$$

is a degree-preserving sequence of maps, and we denote by $(\#)\widehat{\ }$ *the sequence induced by applying the graded-completion operation, then* $(\#)$ *is exact if and only if* $(\#)\widehat{\ }$ *is exact.*

From these facts, we can conclude the following proposition and corollary relating the functors $\widehat{R} \otimes -$ and $(-)\widehat{\ }$.

Proposition 4.2.3. *Let* M *be a* $\mathbb{Z}\frac{1}{s}$*-graded module over a* $\mathbb{Z}\frac{1}{s}$*-graded ring* R*. Then* $\widehat{M} \cong \widehat{R} \otimes M$*.*

Proof. The claim is clearly true for $M = R$ and thus follows for the case where M is a free module as well. For a general graded module M, let $H \to G \to M \to 0$ be a degree-preserving free presentation of M. We then have a commutative diagram:

$$\begin{tikzcd} \widehat{R} \otimes H \arrow[r] \arrow[d] & \widehat{R} \otimes G \arrow[r] \arrow[d] & \widehat{R} \otimes M \arrow[r] \arrow[d] & 0 \\ \widehat{H} \arrow[r] & \widehat{G} \arrow[r] & \widehat{M} \arrow[r] & 0 \end{tikzcd}$$

where the top row is exact by the right exactness of tensor, and the bottom row is exact by (f) of the last lemma. Since H and G are free modules, the first two vertical arrows are isomorphisms. By the Five Lemma ([Mac, Lemma 3.3]), the map $\widehat{R} \otimes M \to \widehat{M}$ is also an isomorphism. □

Corollary 4.2.4. *If* $M_\bullet$ *is an exact sequence of* $\mathbb{Z}\frac{1}{s}$*-graded* R*-modules, then* $\widehat{R} \otimes M_\bullet$ *is also exact.*

4.3 Building a Graded-Complete Algebra That Captures Tight Closure

Let R be an $\mathbb{N}$-graded Noetherian domain of characteristic $p > 0$ with $R_0 = K$ and $m = \bigoplus_{i \geq 1} R_i$. We want to construct an $R^{+\mathrm{gr}}$-algebra B which is $\mathbb{N}$-graded-complete, a graded big Cohen-Macaulay R-algebra, and where contracted-expansion is tight closure. We start by constructing an $\mathbb{N}$-graded $R^{+\mathrm{gr}}$-algebra for which all homogeneous pieces of elements in R that are in a tight closure $(N_m)^*_{M_m}$ are forced into satisfying certain relations which are the homogeneous pieces of relations that force the element into the image of $B \otimes N$ in $B \otimes M$. Then we will enlarge this to an $\mathbb{N}$-graded big Cohen-Macaulay R-algebra by trivializing all relations on all homogeneous systems of parameters in R. Finally, we apply the graded completion operation described above so that we can sum up all of the homogeneous pieces of the tight closure forcing relations.

For the remainder of the section, let S be an $\mathbb{N}$-graded (not necessarily Noetherian) R-algebra. In later sections, we will concentrate on the case that $S = R^{+\mathrm{gr}}$.

Let $\mathrm{TCrel}(R)$ be the set of all $\mu = (M, N, u, \alpha, \nu, \rho)$ such that $M = R^\nu$ is a finitely generated free R-module, $N \subseteq M$ is the submodule generated by the column space of the $\nu \times \rho$ matrix α, and $u \in (N_m)^*_{M_m} \cap M$, where u is given by a $\nu \times 1$ column matrix.

For all $\mu \in \mathrm{TCrel}(R)$, $1 \le j \le \rho$, and $h \in \mathbb{N}$, let $b_{\mu jh}$ be an indeterminate over S of degree h. Let u_i, for $1 \le i \le \nu$, be the entries of u, and let a_{ij} be the entries of α for $1 \le i \le \nu$, $1 \le j \le \rho$. Since these are elements of R, we may write each as a sum of graded pieces:

$$u_i = \sum_{d=0}^{\infty} u_{id} \qquad \text{and} \qquad a_{ij} = \sum_{k=0}^{\infty} a_{ijk},$$

where all but finitely many of the u_{id} and a_{ijk} are zero. Now, let

$$\xi_{\mu id} := u_{id} - \sum_{j=1}^{\rho} \sum_{k+h=d} a_{ijk} b_{\mu jh}$$

be a homogeneous element in $S[b_{\mu jh} : \mu, j, h]$. Let $\mathcal{I}$ be the homogeneous ideal generated by all $\xi_{\mu id}$ in $S[b_{\mu jh} : \mu, j, h]$, for $\mu \in \mathrm{TCrel}(R)$, $1 \le i \le \nu$, and $d \in \mathbb{N}$.

Then

$$\mathrm{TC}(S/R) := S[b_{\mu jh} : \mu, j, h]/\mathcal{I}$$

is an $\mathbb{N}$-graded S-algebra. Since R is usually understood, we will often write $\mathrm{TC}(S)$ for $\mathrm{TC}(S/R)$. We can now form the graded-completion $\widehat{\mathrm{TC}}(S) := (\mathrm{TC}(S))\widehat{\ }$.

The equation $\sum_{d=0}^{\infty} \xi_{\mu id} = 0$ is then forced to hold in $\widehat{\mathrm{TC}}(S)$, which yields

$$u_i = \sum_{j=1}^{\rho} a_{ij} (\sum_{h=0}^{\infty} b_{\mu jh}) \qquad \forall i.$$

Therefore, for $1 \leq j \leq \rho$, the $\sum_{h=0}^{\infty} b_{\mu jh}$ give a solution to the matrix equation $\alpha X = u$ in $\widehat{\mathrm{TC}}(S)$.

Remark 4.3.1. If T is any lim-graded-complete $\widehat{\mathrm{TC}}(S)$-algebra such that $\widehat{\mathrm{TC}}(S) \to T$ is degree-preserving, then we can also solve $\alpha X = u$ in T using the images of the $b_{\mu jh}$. By our construction, the inclusion $(N_m)^*_{M_m} \cap M \subseteq NT \cap M$ is then forced for any such T.

4.4 Building a Graded Big Cohen-Macaulay Algebra

We now introduce the constructions needed to build a possibly improper graded big Cohen-Macaulay algebra for R from a given $\mathbb{N}$-graded R-algebra S. We will force all relations on parameters to become trivial after a countable number of modifications of S. In the next section, we will show that for $S = \widehat{\mathrm{TC}}(R^{+\mathrm{gr}}/R)$ this construction yields a proper graded big Cohen-Macaulay algebra for R. The process will closely follow the methods used in [HH7, Section 3]. We will use similar notation to that of Hochster and Huneke, but we will underline items to distinguish our graded constructions from their local constructions.

For this section, R is any $\mathbb{N}$-graded Noetherian ring of Krull dimension D. For any $\mathbb{N}$-graded R-algebra S, we define $\underline{\mathrm{Rel}}(S)$ to be the set of all $\lambda = (k, \mathbf{x}, \mathbf{s})$ such that $0 \leq k \leq D$, $\mathbf{x} = (x_1, \ldots, x_{k+1})$ is part of a homogeneous system of parameters for R, and $\mathbf{s} = (s_1, \ldots, s_{k+1})$ is a sequence of homogeneous elements in S such that

$$\sum_{i=1}^{k} x_i s_i = x_{k+1} s_{k+1}$$

is a homogeneous relation in S. For $\lambda \in \underline{\mathrm{Rel}}(S)$ and $1 \leq j \leq k$, let $z_{\lambda j}$ be an indeterminate over S with $\deg z_{\lambda j} := \deg s_{k+1} - \deg x_j$ in $\mathbb{Z}$. For all λ, let

$$\theta_\lambda := \sum_{j=1}^{k} x_j z_{\lambda j} - s_{k+1}$$

be a homogeneous element in $S[z_{\lambda j} : \lambda, j]$. Let $\mathcal{J}(\mathcal{S})$ be the homogeneous ideal of $S[z_{\lambda j} : \lambda, j]$ generated by all θ_λ and by all $z_{\lambda j}$ such that $\deg z_{\lambda j} \leq 0$. Now set

$$\underline{\mathrm{Mod}}(S) := S[z_{\lambda j} : \lambda, j]/\mathcal{J}(\mathcal{S}),$$

an $\mathbb{N}$-graded S-algebra.

Let $\underline{\mathrm{Mod}}_1(S) := \underline{\mathrm{Mod}}(S)$. Given $\underline{\mathrm{Mod}}_n(S)$, define $\underline{\mathrm{Mod}}_{n+1}(S) := \underline{\mathrm{Mod}}(\underline{\mathrm{Mod}}_n(S))$. Then each $\underline{\mathrm{Mod}}_n(S)$ is an $\mathbb{N}$-graded S-algebra. Since there is a natural degree-preserving S-algebra homomorphism $\underline{\mathrm{Mod}}_n(S) \to \underline{\mathrm{Mod}}_{n+1}(S)$ for all n, we can define $\underline{\mathrm{Mod}}_\infty(S) := \varinjlim_n \underline{\mathrm{Mod}}_n(S)$, which is also an $\mathbb{N}$-graded S-algebra that preserves the degrees of S.

Proposition 4.4.1. $\underline{\mathrm{Mod}}_\infty(S)$ *is a possibly improper graded big Cohen-Macaulay algebra for* R*.*

Proof. Given a homogeneous relation in $\underline{\mathrm{Mod}}_\infty(S)$ on a homogeneous system of parameters from R, this must also be a relation in some $\underline{\mathrm{Mod}}_n(S)$. By our construction, this relation becomes trivial in $\underline{\mathrm{Mod}}_{n+1}(S)$ and so in $\underline{\mathrm{Mod}}_\infty(S)$. $\square$

Furthermore, we can form the graded-completions $\underline{\mathrm{M\hat{o}d}}_n(S) := (\underline{\mathrm{Mod}}_n(S))\widehat{\ }$ for all $n \in \mathbb{N}$ and $(\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$. The degree-preserving map $\underline{\mathrm{Mod}}_n(S) \to \underline{\mathrm{Mod}}_{n+1}(S)$ induces a degree-preserving map $\underline{\mathrm{M\hat{o}d}}_n(S) \to \underline{\mathrm{M\hat{o}d}}_{n+1}(S)$ for all n that gives a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \underline{\mathrm{M\hat{o}d}}_n(S) \arrow[r, "\widehat{\phi}_n"] & \underline{\mathrm{M\hat{o}d}}_{n+1}(S) \arrow[r] & \cdots \\ \cdots \arrow[r] & \underline{\mathrm{Mod}}_n(S) \arrow[r, "\phi_n"] \arrow[u] & \underline{\mathrm{Mod}}_{n+1}(S) \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

as in Lemma 4.1.4(b). Hence, we may now define the lim-graded-complete ring

$$\underline{\mathrm{M\hat{o}d}}_\infty(S) := \varinjlim_n \underline{\mathrm{M\hat{o}d}}_n(S).$$

From Lemma 4.1.9 we obtain a degree-preserving S-algebra map

$$\Psi : \underline{\mathrm{M\hat{o}d}}_\infty(S) \to (\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }.$$

The utility of these two lim-graded-complete R-algebras is that they are both possibly improper graded big Cohen-Macaulay R-algebras. Indeed, $\underline{\mathrm{M\hat{o}d}}_\infty(S)$ is a direct limit of the $\underline{\mathrm{M\hat{o}d}}_n(S)$, where the direct limit of the $\underline{\mathrm{Mod}}_n(S)$ is $\underline{\mathrm{Mod}}_\infty(S)$, and $(\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$ is a graded-completion of $\underline{\mathrm{Mod}}_\infty(S)$. So, by Proposition 4.4.1 and Lemmas 4.1.15 and 4.1.14, respectively, we obtain the desired result:

Proposition 4.4.2. *The lim-graded-complete S-algebras $\underline{\mathrm{M\hat{o}d}}_\infty(S)$ and $(\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$ are possibly improper graded big Cohen-Macaulay R-algebras.*

Remark 4.4.3. If R is $\mathbb{N}$-graded and $R_0 = K$, then they will be proper as long as the expansion of $m = \bigoplus_{i \geq 1} R_i$ is not the unit ideal. Since we have a notion of degree in the algebras above, this can only happen if $1 = 0$. In order to show that this does not happen in either case, it is sufficient to show it for $(\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$ since it is a degree-preserving $\underline{\mathrm{M\hat{o}d}}_\infty(S)$-algebra. By Lemma 4.1.4(a), $1 = 0$ in $(\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$ if and only if $1 = 0$ in $\underline{\mathrm{Mod}}_\infty(S)$.

In the next section we will show that if $S = R^{\mathrm{TC}}$, then $1 \neq 0$ in $\underline{\mathrm{Mod}}_\infty(S)$ by showing it is true after trivializing a finite number of relations. To show this is sufficient, we need to discuss the notion of *graded algebra modifications* which are a graded analogue of the algebra modifications defined by Hochster and Huneke.

If S is any $\mathbb{N}$-graded R-algebra and $(k, \mathbf{x}, \mathbf{s}) \in \underline{\mathrm{Rel}}(S)$, then we can form the $\mathbb{N}$-graded S-algebra

$$S^{(1)} := S[z_1, \cdots, z_k]/J,$$

where the z_t are indeterminates of $\deg z_t := \deg s_{k+1} - \deg x_t$ and J is the homogeneous ideal generated by all $\sum_{t=1}^{k} x_t z_t - s_{k+1}$ and by all z_t such that $\deg z_t \leq 0$. As in the local case, we call $S^{(1)}$ an $\mathbb{N}$-*graded algebra modification* of S over R. We can then also look at a finite sequence of graded algebra modifications:

$$S = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(h)},$$

where for all $1 \leq i \leq h$, $S^{(i+1)}$ is a $\mathbb{N}$-graded algebra modification of $S^{(i)}$ over R. We will make use of the following graded analogue of Proposition 2.3.8 and [HH7, Proposition 3.7]:

Proposition 4.4.4. *Let S be an $\mathbb{N}$-graded R-algebra.*

(a) *If $S = S^{(0)} \to S^{(1)} \to \cdots \to S^{(h)}$ is a finite sequence of $\mathbb{N}$-graded algebra modifications of S over R, then there exists a degree-preserving S-algebra map*

$$S^{(h)} \to \underline{\mathrm{Mod}}_h(S),$$

and thus a degree-preserving S-algebra map $S^{(h)} \to \underline{\mathrm{Mod}}_\infty(S)$.

(b) *Given any $\mathbb{N}$-graded S-algebra S' and a degree-preserving map $S' \to \underline{\mathrm{Mod}}_\infty(S)$ of S-algebras, there exists a finite sequence of $\mathbb{N}$-graded algebra modifications $S = S^{(0)} \to S^{(1)} \to \cdots \to S^{(h)}$ of S over R and a degree-preserving S-algebra map $S' \to S^{(h)}$.*

(c) *$1 \neq 0$ in $\underline{\mathrm{Mod}}_\infty(S)$ if and only if for every finite sequence of $\mathbb{N}$-graded algebra modifications $S = S^{(0)} \to S^{(1)} \to \cdots \to S^{(h)}$ of S over R, $1 \neq 0$ in $S^{(h)}$.*

Therefore, in order to establish that $\underline{\mathrm{Mod}}_{\infty}(S)$ and $(\underline{\mathrm{Mod}}_{\infty}(S))\widehat{}$ are proper graded big Cohen-Macaulay R-algebras, it will suffice to show that $1 \neq 0$ in $S^{(h)}$ for all finite sequences of $\mathbb{N}$-graded algebra modifications $S = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(h)}$.

4.5 The Existence Proof

Let R be an $\mathbb{N}$-graded Noetherian domain of characteristic $p > 0$ such that $R_0 = K$. Our hypotheses on R then guarantee the existence of a completely stable homogeneous test element c of degree $\delta > 0$ by Theorem 2.2.7 from Hochster and Huneke.

Put $S^{(-1)} := R^{+\mathrm{gr}}$ and

$$S^{(0)} := \mathrm{TC}(R^{+\mathrm{gr}}) = R^{+\mathrm{gr}}[b_{\mu jh} : \mu, j, h]/(\xi_{\mu id} : \mu, i, d)$$

as defined in the third section. Choose some ordering on the $b_{\mu jh}$ and write them as $z_t^{(-1)}$ for $t \in \mathbb{N}$. Given $S^{(i)}$ for $0 \leq i \leq h-1$, let

$$S^{(i+1)} := S^{(i)}[z_1^{(i)}, \ldots, z_{k_i}^{(i)}]/J_i$$

such that J_i is the ideal generated by all $s^{(i)} - \sum_{t=1}^{k_i} x_t^{(i)} z_t^{(i)}$ and by all $z_t^{(i)}$ such that $\deg z_t^{(i)} \leq 0$, where $x_1^{(i)}, \cdots, x_{k_i}^{(i)}, x_{k_i+1}^{(i)}$ is a homogeneous system of parameters in R,

$$x_{k_i+1}^{(i)} s^{(i)} = \sum_{t=1}^{k_i} x_t^{(i)} s_t^{(i)}$$

is a homogeneous relation in $S^{(i)}$, and the $z_t^{(i)}$ are indeterminates with $\deg z_t^{(i)} := \deg s^{(i)} - \deg x_t^{(i)}$. Then

$$\mathrm{TC}(R^{+\mathrm{gr}}) = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(h)}$$

is a finite sequence of graded algebra modifications.

We need to introduce a few new rings at this point. First, by a simple generalization of [HH3, Lemma 4.1, Proposition 4.2], which define $R^{+\mathrm{gr}}$, we can define

$$T_q := (R^{1/q})^{+\mathrm{gr}}$$

as the $\mathbb{N}\frac{1}{q}$-graded subring of $(R^{1/q})^+ \cong R^+$. We can further use Theorem 2.3.4 to see that T_q is a graded big Cohen-Macaulay algebra for $R^{1/q}$ and then also for R since every homogeneous system of parameters in R is a homogeneous system of parameters in $R^{1/q}$. Then

$$U_q := T_q[c^{-1/q}]$$

is a $\mathbb{Z}\frac{1}{q}$-graded $R^{+\mathrm{gr}}$-algebra. So, $\widehat{T}_q$ is an $\mathbb{N}\frac{1}{q}$-graded-complete ring, and $\widehat{U}_q$ is a $\mathbb{Z}\frac{1}{q}$-graded-complete ring. Additionally, the degree-preserving inclusion $T_q \hookrightarrow U_q$, where $\deg c^{-1/q} := -(1/q)\deg c$, implies that $\widehat{T}_q \hookrightarrow \widehat{U}_q$ as $\mathbb{Z}\frac{1}{q}$-graded-complete rings by Lemma 4.1.4(c). By Lemma 4.1.14, we have the following helpful fact:

Proposition 4.5.1. *For all q, $\widehat{T}_q$ is a degree-preserving graded big Cohen-Macaulay R-algebra.*

For sufficiently large q we will construct inductively degree-preserving maps of $R^{+\mathrm{gr}}$-algebras from each $S^{(i)}$ to $\widehat{U}_q$. The existence of the map $S^{(h)} \to \widehat{U}_q$ will show that $1 \neq 0$ in $S^{(h)}$ as this is true in U_q and so in $\widehat{U}_q$.

In order to construct the maps we will need to keep track of certain numerical bounds associated with the images of selected elements of each $S^{(i)}$. It is important that these bounds be independent of q. For all $0 \leq j \leq h-1$, we will use reverse induction to define Γ_j (a finite subset of $S^{(j)}$ of homogeneous elements) and positive integers $b(j)$ and $B(j)$.

First, let

$$\Gamma_{h-1} := \{s^{(h-1)}, s_1^{(h-1)}, \cdots, s_{k_{h-1}}^{(h-1)}\}.$$

Now, given Γ_j, each element can be written as a homogeneous polynomial in the $z_t^{(j-1)}$ with homogeneous coefficients in $S^{(j-1)}$. Let $b(j)$ be the largest degree of any such polynomial. Let Γ_{j-1} be the set of all coefficients of these polynomials together with $s^{(j-1)}, s_1^{(j-1)}, \cdots, s_{k_{j-1}}^{(j-1)}$. Now define $B(-1) := 1$, $B(0) := b(0)$, and given $B(j)$ for $0 \leq j \leq h-2$, let

$$B(j+1) := B(j)(b(j+1)+1).$$

Notice that, as claimed, all $B(j)$ are independent of q.

Start with $q \geq q(0) := 1$. To define a degree-preserving $R^{+\mathrm{gr}}$-algebra map

$$\psi_q^{(0)} : S^{(0)} \to \widehat{U}_q,$$

we need to find a homogeneous image for each $b_{\mu jh}$ such that each

$$\xi_{\mu id} = u_{id} - \sum_{j=1}^{\rho} \sum_{k+h=d} a_{ijk} b_{\mu jh}, \tag{4.5.2}$$

maps to zero. Given $(M, N, u, \alpha, \nu, \rho) \in \mathrm{TCrel}(R)$, $u \in (N_m)^*_{M_m} \cap M$ implies that

$$c^{1/q} u = \alpha X$$

has a solution in $(R_m)^{1/q} \subseteq \widehat{R}^{1/q}$ (see Lemma 4.1.3), say $X_j = r_j^{1/q}$, where $r_j \in \widehat{R}$, for $1 \leq j \leq \rho$. Write each $r_j = \sum_{h=0}^{\infty} r_{jh}$ such that $\deg r_{jh} = h$ in $\widehat{R}$. Then $r_j^{1/q} = \sum_h r_{jh}^{1/q}$, where $\deg r_{jh}^{1/q} = h/q$ in $\widehat{R}^{1/q}$. If the entries of u are u_i and the entries of α are a_{ij}, then

$$c^{1/q} u_i = \sum_{j=1}^{\rho} a_{ij} r_j^{1/q}. \tag{4.5.3}$$

As $R^{1/q} \subseteq (R^{1/q})^{+\mathrm{gr}} = T_q$ and $\widehat{R}^{1/q} \cong (R^{1/q})\widehat{\ }$, by Lemma 4.1.4(c), $\widehat{R}^{1/q} \hookrightarrow \widehat{T}_q$ so that (4.5.3) holds in $\widehat{T}_q$ too.

Taking homogeneous pieces, we have

$$c^{1/q} u_{id} = \sum_{j=1}^{\rho} \sum_{k+h/q=\Delta_d} a_{ijk} r_{jh}^{1/q} \tag{$\#_{id}$}$$

for all $1 \le i \le \nu$ and for all $d \in \mathbb{N}$, where $\deg u_{id} = d$, $\deg a_{ijk} = k$, and $\Delta_d := d + \delta/q$. Since $\widehat{T}_q \hookrightarrow \widehat{U}_q$ while preserving degree, $(\#_{id})$ holds in $\widehat{U}_q$ as well. Hence, in $\widehat{U}_q$ we have

$$u_{id} = \sum_{j=1}^{\rho} \sum_{k+h/q=\Delta_d} a_{ijk}(c^{-1/q} r_{jh}^{1/q}) \tag{4.5.4}$$

for all $1 \le i \le \nu$ and $d \in \mathbb{N}$. Define

$$\psi_q^{(0)}(b_{\mu jh}) := c^{-1/q} r_{jh}^{1/q}.$$

Then all $\xi_{\mu id}$ will map to zero, and $\psi_q^{(0)}$ will be well-defined. Moreover, since (4.5.2) and (4.5.4) are homogeneous,

$$\deg c^{-1/q} r_{jh}^{1/q} = \deg u_{id} - \deg a_{ijk} = \deg b_{\mu jh}$$

so that $\psi_q^{(0)}$ is degree-preserving. Finally, each $b_{\mu jh}$ maps to $c^{-1/q}\widehat{T}_q = c^{-B(-1)/q}\widehat{T}_q$, and Γ_0 maps to $c^{-b(0)/q}\widehat{T}_q = c^{-B(0)/q}\widehat{T}_q$ as the elements of Γ_0 can all be written as polynomials of degree $\le b(0)$ in the $b_{\mu jh}$.

Suppose that for some $0 \le i \le h-1$ and all $q \ge q(i)$ we have a degree-preserving $R^{+\mathrm{gr}}$-algebra map $\psi_q^{(i)} : S^{(i)} \to \widehat{U}_q$, where the $z_t^{(i-1)}$ all map to $c^{-B(i-1)/q}\widehat{T}_q$, and Γ_i maps to $c^{-B(i)/q}\widehat{T}_q$. We will extend $\psi_q^{(i)}$ for $q \gg 0$ to a degree-preserving map from $S^{(i+1)}$. If $i \le h-2$, we will also map each $z_t^{(i)}$ to $c^{-B(i)/q}\widehat{T}_q$, and map Γ_{i+1} to $c^{-B(i+1)/q}\widehat{T}_q$.

In order to simplify notation, we drop many of the (i) labels on parameters. Then

$$S^{(i+1)} = S^{(i)}[z_1, \ldots, z_k]/J,$$

where J is the ideal generated by $s - \sum_{t=1}^{k} x_t z_t$ and all z_t such that $\deg z_t \le 0$. Since s and the s_t (in the relation $x_{k+1}s = \sum_{t=1}^{k} x_t s_t$ in $S^{(i)}$) are in Γ_i, we can write

$$\psi_q^{(i)}(s) = c^{-B(i)/q}\sigma \qquad \text{and} \qquad \psi_q^{(i)}(s_t) = c^{-B(i)/q}\sigma_t \tag{4.5.5}$$

for all $q \geq q(i)$, where σ and the σ_t are homogeneous elements of $\widehat{T}_q$. Hence,

$$x_{k+1}\psi_q^{(i)}(s) = \sum_{t=1}^{k} x_t \psi_q^{(i)}(s_t)$$

in $\widehat{U}_q$, and multiplying through by $c^{B(i)/q}$ yields

$$x_{k+1}\sigma = \sum_{t=1}^{k} x_t \sigma_t$$

in $\widehat{T}_q$ (as $\widehat{T}_q$ injects into $\widehat{U}_q$). By Proposition 4.5.1, $\widehat{T}_q$ is a graded big Cohen-Macaulay algebra over R. Therefore,

$$\sigma = \sum_{t=1}^{k} x_t \tau_t, \tag{4.5.6}$$

where $\tau_t = 0$ or τ_t is a homogeneous element in $\widehat{T}_q$ with $\deg \tau_t = \deg \sigma - \deg x_t$. Thus, from (4.5.5) and (4.5.6) we have $\psi_q^{(i)}(s) = \sum_{t=1}^{k} x_t (c^{-B(i)/q}\tau_t)$ in $\widehat{U}_q$.

If $\deg z_t \leq 0$, then we see from (4.5.5) that

$$\begin{aligned} (\deg \sigma - (B(i)/q)\deg c) - \deg x_t &= \deg \psi_q^{(i)}(s) - \deg x_t \\ &= \deg s - \deg x_t = \deg z_t \leq 0. \end{aligned} \tag{4.5.7}$$

This implies that

$$\deg \sigma \leq \deg x_t + (B(i)/q)\deg c.$$

Since $\deg c$ is positive and $B(i)$ is independent of q, we can find $q(i+1) \geq q(i)$ such that

$$(B(i)/q)\deg c < 1 \tag{4.5.8}$$

for all $q \geq q(i+1)$. We then have $\deg \sigma < \deg x_t + 1$, i.e., $\deg \sigma \leq \deg x_t$, for all such t and all $q \geq q(i+1)$. Since $\sigma = \sum_{t=1}^{k} x_t \tau_t$ in $\widehat{T}_q$, which only has non-negative degrees, either $\tau_t = 0$ or $\deg \sigma \geq \deg x_t$ for all t. Hence we see that for all t such that $\deg z_t \leq 0$ either $\tau_t = 0$ or $\deg \sigma = \deg x_t$ when $q \geq q(i+1)$. If $\deg \sigma = \deg x_t$, however, then by (4.5.7) and (4.5.8) we see $\deg z_t = -(B(i)/q)\deg c > -1$, but $\deg z_t$ is assumed to be a non-positive integer, so $\deg z_t = 0$. This implies that $\deg c = 0$ too, a contradiction. Therefore, if $\deg z_t \leq 0$, then $\tau_t = 0$ for all $q \geq q(i+1)$.

All of this now implies that for $q \geq q(i+1)$, there is a well-defined $S^{(i)}$-algebra map $\psi_q^{(i+1)} : S^{(i+1)} \to \widehat{U}_q$ extending $\psi_q^{(i)}$ given by

$$\psi_q^{(i+1)}(z_t) = c^{-B(i)/q}\tau_t.$$

From (4.5.6) and (4.5.7), if $\tau_t \neq 0$, then we have

$$\deg z_t = \deg \sigma - (B(i)/q)\deg c - \deg x_t = \deg \tau_t - (B(i)/q)\deg c,$$

so that our map is also degree-preserving. If, in addition, $i \leq h-2$, then the z_t map to $c^{-B(i)/q}\widehat{T}_q$, and Γ_{i+1} maps to $c^{-B(i+1)/q}\widehat{T}_q$, since $B(i+1) = B(i)b(i+1) + B(i)$ and these elements can be written as polynomials in the z_t of degree at most $b(i+1)$ with coefficients in Γ_i.

We can finally conclude that there exists a degree-preserving $R^{+\mathrm{gr}}$-algebra map

$$\psi_q^{(h)} : S^{(h)} \to \widehat{U}_q$$

for all $q \gg 0$.

Therefore $1 \neq 0$ in $S^{(h)}$ which, by Proposition 4.4.4, shows that $1 \neq 0$ in $\underline{\mathrm{Mod}}_\infty(\mathrm{TC}(R^{+\mathrm{gr}}))$. Combining this result with Proposition 4.4.2 and Remark 4.4.3, we have the following:

Theorem 4.5.9. $\underline{\mathrm{M\hat{o}d}}_\infty(\mathrm{TC}(R^{+\mathrm{gr}}))$ *and* $(\underline{\mathrm{Mod}}_\infty(\mathrm{TC}(R^{+\mathrm{gr}})))\widehat{\ }$ *are both graded big Cohen-Macaulay R-algebras.*

4.6 Consequences

If $S = \mathrm{TC}(R^{+\mathrm{gr}})$, where R is $\mathbb{N}$-graded Noetherian domain and $R_0 = K$, then let $B = \underline{\mathrm{M\hat{o}d}}_\infty(S)$ or $B = (\underline{\mathrm{Mod}}_\infty(S))\widehat{\ }$. By construction, B is a degree-preserving lim-graded-complete $\mathrm{TC}(R^{+\mathrm{gr}})$-algebra and thus $(N_m)^*_{M_m} \cap M \subseteq NB \cap M$ for all free R-modules M and $N \subseteq M$ by Remark 4.3.1.

By Lemmas 4.1.3 and 4.1.11, $\widehat{R}$ is a complete local domain, and by our construction of B, we can see that B is an $\widehat{R}$-algebra. A homogeneous system of parameters for R is a system of parameters for R_m, and so is a system of parameters for $\widehat{R}$ as $\widehat{R}$ is faithfully flat over R_m. Hence, there exists a system of parameters for $\widehat{R}$ that is a regular sequence on B. By [Ho3, Proposition 10.5], B is solid over $\widehat{R}$. Since R has a completely stable test element (see Theorem 2.2.7), so does $\widehat{R}$. Thus, tight closure equals solid closure in $\widehat{R}$ by Theorem 2.5.4, and by [HH1, Proposition 8.13c], $(N_m)^* = ((N_m)\widehat{\ })^* \cap M_m$. Therefore,

$$NB \cap M \subseteq (N_m)\widehat{\ }B \cap M \subseteq ((N_m)\widehat{\ })^{\bigstar} \cap M = ((N_m)\widehat{\ })^* \cap M = (N_m)^* \cap M,$$

where the second inclusion follows because B is solid over $\widehat{R}$. We conclude the following:

Corollary 4.6.1. *For $N \subseteq M$, where M is free over R, we have*

$$(N_m)^*_{M_m} \cap M = NB \cap M.$$

For any pair of R-modules $N \subseteq M$ with $u \in M$, we have $u/1 \in (N_m)^*_{M_m} \cap M$ if and only if $1 \otimes u \in \mathrm{Im}(B \otimes N \to B \otimes M)$.

The second claim follows because when we compute tight closure for a pair of modules, we can always assume the ambient module is free by applying Proposition 2.2.3(f). If, furthermore, M is free and all $N_M^{[q]}$ are contracted from M_m (e.g., M/N is m-coprimary), then [HH1, Proposition 8.9] implies that $(N_m)^*_{M_m} \cap M = N^*_M$, which implies that $N^*_M = NB \cap M$ in these cases.

CHAPTER 5

A Study of Solid Algebras

Due to their strong connections to tight closure theory, solid algebras continue to be an important subject of study. In this chapter we will concentrate on Hochster's result, Theorem 2.5.5, which says that over a complete local domain, any algebra that maps to a big Cohen-Macaulay algebra is solid. The converse question of whether all solid algebras, in characteristic p, map to big Cohen-Macaulay algebras remains open, in general.

We will further investigate solid algebras in the hope that this study will eventually produce an answer to whether solid algebras over a complete local domain in positive characteristic are exactly the algebras that map to big Cohen-Macaulay algebras. Our first attempt is a direct approach that tries to exploit the local cohomology criterion for solidity over a complete local domain (see Proposition 2.5.2(d)) to show that every algebra modification of a solid algebra is a solid algebra, as this result will answer the question.

The second approach is a characterization of solid algebras in terms of *phantom extensions* (see Section 5.2.1 or [HH5, Section 5]) and direct limits of phantom extensions. Despite this new characterization of solid algebras, the question, "Do solid algebras map to big Cohen-Macaulay algebras?" remains open in characteristic p.

5.1 Solid Algebras and Big Cohen-Macaulay Algebras

In this section, we investigate the following question:

Question 5.1.1. In positive characteristic, does a solid algebra over a complete local domain map to a big Cohen-Macaulay algebra?

The converse of this statement is true in any characteristic, as shown in Theorem 2.5.5. Hochster also shows that in equal characteristic 0, the answer is "no" in general (see [Ho3, Example 10.7]), but the answer is "yes" in any characteristic when $\dim R \leq 2$ (see [Ho3, Theorem 12.5]). The general answer in mixed characteristic and in positive characteristic is unknown. We will study the question in positive characteristic.

5.1.1 Reductions of the Problem

Let R be a complete local domain of positive characteristic p and dimension n, and let S be a solid R-algebra. If $x_1, \ldots, x_{h+1}$ is part of a system of parameters in R, and $s_1, \ldots, s_{h+1}$ are elements in S such that $x_1 s_1 + \cdots + x_{h+1} s_{h+1} = 0$, then recall from Chapter 2 that

$$T = S[U_1, \ldots, U_h]/(s_{h+1} - \sum_{i=1}^{h} x_i U_i)$$

is an algebra modification of S over R. Using this notion, we also record the following question originally posed by Hochster in [Ho3]:

Question 5.1.2. In positive characteristic, is every algebra modification of a solid algebra over a complete local domain still solid?

Hochster also points out that our two questions have equivalent answers.

Lemma 5.1.3 (Hochster, [Ho3])**.** *For any fixed complete local domain, Questions 5.1.1 and 5.1.2 are equivalent.*

Proof. Since the proof is brief, we will reproduce it here. Suppose Question 5.1.1 has a positive answer. If S is solid over a complete local domain (R, m), then S maps to a big Cohen-Macaulay algebra B. If T is an algebra modification of S over R, then T also maps to B. By Theorem 2.5.5, T is solid.

If we instead suppose that Question 5.1.2 can be answered positively, then every finite sequence of algebra modifications of a solid algebra S terminates in a solid algebra T. Therefore, $1 \notin mT$ (using the local cohomology criterion), and so Proposition 2.3.8 implies that S can be mapped to a big Cohen-Macaulay algebra. $\square$

While investigating Question 5.1.2, we will show that we can make several reductions to simplify the problem. We will show that one can assume R is a formal power series ring over a field, that S is a finitely generated R-algebra domain, and that the algebra modification is constructed with respect to a full system of parameters

Lemma 5.1.4. *In order to prove that Question 5.1.2 has an affirmative answer, it suffices to assume* $R = K[[x_1, \ldots, x_n]]$.

Proof. Since R is a complete local domain containing a field, R is a module-finite extension ring of $A = K[[x_1, \ldots, x_n]]$, where we have extended the partial system of parameters $x_1, \ldots, x_{k+1}$ (used to construct the algebra modification T of the solid algebra S) to a full system of parameters $x_1, \ldots, x_n$. By Proposition 2.5.2(c), S is solid over A, and T is solid over R if and only if it is solid over A. Furthermore, T is also an algebra modification of S over A. $\square$

If (R, m) is a complete local domain of dimension n with $x_1, \ldots, x_n$ any system of parameters, then, by the local cohomology criterion, S is solid over R if and only if $H^n_m(S) \neq 0$. In positive characteristic, we can then deduce the following computational result.

Lemma 5.1.5. *Let (R, m) be a positive characteristic complete local domain of dimension $n \geq 1$ with system of parameters $x_1, \ldots, x_n$. An R-algebra S is solid over R if and only if $(x_1 \cdots x_n)^k \notin (x_1^{k+1}, \ldots, x_n^{k+1})S$ for all $k \geq 0$.*

Proof. The necessity comes from [Ho3, Observation 2.6]. For the converse, suppose $(x_1 \cdots x_n)^k$ is in $(x_1^{k+1}, \ldots, x_n^{k+1})S$, for some k. Then for all $q = p^e$,

$$(x_1 \cdots x_n)^{kq} \in (x_1^{kq+q}, \ldots, x_n^{kq+q})S.$$

Let $[z + (x_1^N, \ldots, x_n^N)S] \in H^d_{(x_1,\ldots,x_n)}(S)$. Choose $q \geq N$. Then

$$(x_1 \cdots x_n)^{kq} \in (x_1^{kq+q}, \ldots, x_n^{kq+q})S \subseteq (x_1^{kq+N}, \ldots, x_n^{kq+N})S.$$

Thus,

$$\begin{aligned} [z + (x_1^N, \ldots, x_n^N)S] &= [z(x_1 \cdots x_n)^{kq} + (x_1^{kq+N}, \ldots, x_n^{kq+N})S] \\ &= [0 + (x_1^{kq+N}, \ldots, x_n^{kq+N})S]. \end{aligned}$$

□

Lemma 5.1.6. *In order to answer Question 5.1.2, it suffices to do the case where S is finite type over R.*

Proof. Suppose that Question 5.1.2 has a positive answer when S is a finitely generated R-algebra. Now, let S be an arbitrary solid R-algebra with a nontrivial relation $x_1 s_1 + \cdots + x_{h+1} s_{h+1} = 0$ in S. If

$$T = S[U_1, \ldots, U_h]/(s_{h+1} - \sum_{i=1}^{h} x_i U_i)$$

is an algebra modification that is not solid, then for some k, we have an inclusion $(x_1 \cdots x_n)^k \in (x_1^{k+1}, \ldots, x_n^{k+1})T$. Since elements of T are represented by polynomials in $S[U_1, \ldots, U_h]$, the inclusion above yields

$$(5.1.7) \quad (x_1 \cdots x_n)^k = x_1^{k+1} F_1(U) + \cdots + x_n^{k+1} F_n(U) + \left(s_{h+1} - \sum_{i=1}^{h} x_i U_i \right) W(U),$$

for polynomials F_j and W in $S[U_1, \ldots, U_h]$. Let $f_j^{(t_1,\ldots,t_h)} \in S$ be the coefficient of $U_1^{t_1} \cdots U_h^{t_h}$ in F_j, for all j and similarly define $w^{(t_1,\ldots,t_h)} \in S$ as a coefficient of W (with the convention that $w^{(t_1,\ldots,t_h)} = 0$ if any $t_i < 0$). With these definitions and (5.1.7), we obtain the following system of equations in S:

$$\begin{aligned} & x_1 s_1 + \cdots + x_{h+1} s_{h+1} = 0 \\ & (x_1 \cdots x_n)^k = x_1^{k+1} f_1^{(0,\ldots,0)} + \cdots + x_n f_n^{(0,\ldots,0)} + s_{h+1} w^{(0,\ldots,0)}, \\ & x_1 w^{(t_1-1,t_2,\ldots,t_h)} + \cdots + x_h w^{(t_1,\ldots,t_{h-1},t_h-1)} \\ & \quad = x_1^{k+1} f_1^{(t_1,\ldots,t_h)} + \cdots + x_n f_n^{(t_1,\ldots,t_h)} + s_{h+1} w^{(t_1,\ldots,t_h)} \\ & \quad \forall (t_1, \ldots, t_h) \neq (0, \ldots, 0). \end{aligned} \tag{5.1.8}$$

Since the F_j and W have only finitely many nonzero coefficients, (5.1.8) is a finite system of equations that holds in S.

Let Θ be the set of these equations, and let Σ be the finite set of all s_j, $f_j^{(t_1,\ldots,t_h)}$, and $w^{(t_1,\ldots,t_h)}$. Let

$$S' = R[\Sigma]/(\Theta),$$

and let

$$T' = S'[U_1, \ldots, U_h]/(s_{h+1} - \sum_{i=1}^{h} x_i U_i).$$

Then S' maps to S so that S' is also solid over R. By our assumption, T' is solid since it is an algebra modification of S' over R. Since the equations given by (5.1.8) hold in S', we see that $(x_1 \cdots x_n)^k \in (x_1^{k+1}, \ldots, x_n^{k+1})T'$, a contradiction. $\square$

We can also assume that S is a domain by making use of *minimal solid algebras*. In [Ho3, Section 6], Hochster defines a solid R-algebra as *minimal* if no proper homomorphic image of S is solid. It is clear that if S is a Noetherian solid algebra, then S maps onto a minimal solid algebra. It is also shown in [Ho3, Section 6] that Noetherian minimal solid algebras are domains.

Lemma 5.1.9. *To answer Question 5.1.2, it suffices to do the case where S is a domain.*

Proof. By the last lemma, we may assume that S is of finite type over R, and so also is Noetherian. Since S is Noetherian, it maps onto a minimal solid R-algebra $S/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of S. If T is an algebra modification of S over R, then

$$T' = S/\mathfrak{p}[U_1, \ldots, U_h]/(s_{h+1} - \sum_{i=1}^{h} x_i U_i)$$

is an algebra modification of $S/\mathfrak{p}$ over R. Since T maps to T', T is solid if T' is solid. □

Lemma 5.1.10. *To answer Question 5.1.2, it suffices to do the case where the modification T is a modification of a relation on a full system of parameters*

Proof. By extending the parameters $x_1, \ldots, x_{h+1}$ to a full system $x_1, \ldots, x_n$ and extending $s_1, \ldots, s_{h+1}$ to $s_1, \ldots, s_n$ by setting $s_i = 0$ for all $h + 2 \leq i \leq n$, we can reduce the problem as claimed. □

We have now reduced our problem to the following question: (Notice that $R \hookrightarrow S$ if S is solid because the R-module composition map $R \to S \xrightarrow{\alpha} R$, where $\alpha(1) \neq 0$, is just multiplication by $\alpha(1)$ in the domain R.)

Question 5.1.11. Let $R = K[[x_1, \ldots, x_n]]$, where K is a characteristic $p > 0$ field, and let S be an R-algebra domain such that $R \hookrightarrow S$ and $x_1 s_1 + \cdots + x_n s_n = 0$ in S. If

$$T = S[U_1, \ldots, U_{n-1}]/(s_n - \sum_{i=1}^{n-1} x_i U_i)$$

is an algebra modification of S over R, and if S is solid over R, then is T solid too?

For a moment, we will let R be any complete local domain, with a system of parameters $x_1, \ldots, x_n$, in order to study properties of algebra modifications that are not solid over R. If T is any algebra modification of S over R for the relation $x_1s_1+\cdots+x_ns_n=0$ in S, and T is not solid, then there is a $k\in\mathbb{N}$ and polynomials $F_1,\ldots,F_n,W$ in $S[U_1,\ldots,U_{n-1}]$ such that

$$(x_1\cdots x_n)^k = x_1^{k+1}F_1(U)+\cdots+x_n^{k+1}F_n(U)+\left(s_n-\sum_{i=1}^{n-1}x_iU_i\right)W(U), \tag{5.1.12}$$

as in (5.1.7). Given (5.1.12), there exists a degree bound $d\in\mathbb{N}$ for the F_j and W such that $\deg F_j,\deg W\le d$, for all j.

Definition 5.1.13. Let R be a complete local domain, S be an R-algebra, and T be an algebra modification of S over R. Given (5.1.12) and a degree bound d on the F_j and W, we call T a *hollow modification of type* (k,d).

If T is a hollow modification of S with coefficients of the F_j and W denoted by $f_j^{(t_1,\ldots,t_{n-1})}$ and $w^{(t_1,\ldots,t_{n-1})}$ (resp.) in S, as in (5.1.8), then the following system of equations holds in S.

$$\begin{aligned}
&x_1s_1+\cdots+x_ns_n=0\\
&(x_1\cdots x_n)^k = x_1^{k+1}f_1^{(0,\ldots,0)}+\cdots+x_nf_n^{(0,\ldots,0)}+s_nw^{(0,\ldots,0)},\\
&x_1w^{(t_1-1,t_2,\ldots,t_{n-1})}+\cdots+x_{n-1}w^{(t_1,\ldots,t_{n-2},t_{n-1}-1)}\\
&\quad = x_1^{k+1}f_1^{(t_1,\ldots,t_{n-1})}+\cdots+x_nf_n^{(t_1,\ldots,t_{n-1})}+s_nw^{(t_1,\ldots,t_{n-1})}\\
&\quad \forall(t_1,\ldots,t_{n-1})\neq(0,\ldots,0).
\end{aligned} \tag{5.1.14}$$

If $\deg F_j,\deg W\le d$, S is a domain, and $R\hookrightarrow S$, then we can also see that

$$x_jw^{(0,\ldots,0,d,0,\ldots,0)}=0,$$

where d is in the j^{th} position, and so all $w^{(0,\ldots,0,d,0,\ldots,0)}=0$ under the above conditions.

We will now define a family of finitely generated $\mathbb{Z}/p\mathbb{Z}$-algebras that will help classify when an R-algebra has a hollow modification.

Definition 5.1.15. For $n \geq 1$, let

$$\Sigma^{(n,d)} := \{s_1, \ldots, s_n, f_j^{(t_1,\ldots,t_{n-1})}, w^{(t_1,\ldots,t_{n-1})} \,|\, 1 \leq j \leq n,\ t_1 + \cdots + t_{n-1} \leq d\}$$

be a set of indeterminates over $\mathbb{Z}/p\mathbb{Z}$, and let

$$\Theta^{(n,k,d)} := \{\text{the equations of (5.1.14)}\} \cup \{w^{(0,\ldots,0,d,0,\ldots,0)} \,|\, 1 \leq j \leq n-1\}.$$

We now define

$$A_p^{(n,k,d)} := \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n, \Sigma^{(n,d)}]/(\Theta^{(n,k,d)}),$$

and

$$\widehat{A}_p^{(n,k,d)} := \mathbb{Z}/p\mathbb{Z}[[x_1, \ldots, x_n]][\Sigma^{(n,d)}]/(\Theta^{(n,k,d)})$$

We can now restate the problem in Question 5.1.1 using the algebras $A_p^{(n,k,d)}$.

Proposition 5.1.16. *For a complete local domain R of dimension $n \geq 1$ and positive characteristic p, the following are equivalent:*

(i) *Every solid R-algebra S maps to some big Cohen-Macaulay R-algebra.*

(ii) *No solid R-algebra has a hollow modification.*

(iii) $H^n_{(x_1,\ldots,x_n)}(\widehat{A}_p^{(n,k,d)}) = 0$, *for all k, d.*

(iv) $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,k,d)}) = 0$, *for all k, d.*

(v) *For all k, d, there exists $N = N(k,d)$ such that*

$$(x_1 \cdots x_n)^N \in (x_1^{N+1}, \ldots, x_n^{N+1})A_p^{(n,k,d)}.$$

Proof. The equivalence of (i) and (ii) is given by Lemma 5.1.3. Since every element of $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,k,d)})$ and $H^n_{(x_1,\ldots,x_n)}(\widehat{A}_p^{(n,k,d)})$ is killed by a power of the ideal $(x_1, \ldots, x_n)$, we have $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,k,d)}) \cong H^n_{(x_1,\ldots,x_n)}(\widehat{A}_p^{(n,k,d)})$. Therefore, (iii) is equivalent to (iv). The equivalence of (iv) and (v) can be seen from Lemma 5.1.5.

To show (ii) and (iii) are equivalent, we will use Lemma 5.1.4 to assume that $R = K[[x_1, \ldots, x_n]]$. By construction, $\widehat{A}_p^{(n,k,d)}$ is an R-algebra that has a hollow modification of type (k,d). Thus, if (ii) holds, then $\widehat{A}_p^{(n,k,d)}$ cannot be solid over R, and (iii) follows. If we suppose that (iii) holds, but (ii) does not, then by Lemmas 5.1.9 and 5.1.10, there exists a solid domain S and a hollow modification T of type (k,d) constructed with respect to $x_1,\ldots,x_n$. Therefore the system of equations (5.1.14) holds in S, and so S is an $\widehat{A}_p^{(n,k,d)}$-algebra. Since we assumed that S is solid, $\widehat{A}_p^{(n,k,d)}$ is also solid, but this fact contradicts (iii). Hence, we have finally shown that (iii) implies (ii). $\square$

5.1.2 The Family of Algebras $A_p^{(n,k,d)}$, $n \le 2$

First note that if $\dim R = 0$, then we can assume that R is a field by Lemma 5.1.4. Since all algebras over a field are solid, Question 5.1.1 obviously has a positive answer in dimension 0. When $n = 1$,

$$A_p^{(1,k,d)} = \mathbb{Z}/p\mathbb{Z}[x_1, s_1, f_1]/(x_1^k - x_1^{k+1} f_1)$$

as d is forced to be 0. Clearly, $H^1_{(x_1)}(A_p^{(1,k,d)}) = 0$.

For $n = 2$, we also show that Question 5.1.1 can be answered affirmatively. (This fact is also demonstrated in [Ho3, Section 12] by explicitly constructing a big Cohen-Macaulay algebra to which a given solid algebra maps.) In this case,

$$A_p^{(2,k,d)} = \frac{\mathbb{Z}/p\mathbb{Z}[x_1, x_2, s_1, s_2, f_1^{(t)}, f_2^{(t)}, w^{(t)} \mid 0 \le t \le d]}{\left(\begin{array}{c} x_1 s_1 + x_2 s_2 \\ (x_1x_2)^k - x_1^{k+1} f_1^{(0)} - x_2^{k+1} f_2^{(0)} - s_2 w^{(0)} \\ x_1 w^{(t-1)} - x_1^{k+1} f_1^{(t)} - x_2^{k+1} f_2^{(t)} - s_2 w^{(t)} \quad \forall t \ge 1 \\ w^{(d)} \end{array}\right)} \tag{5.1.17}$$

We will now show that $(x_1x_2)^{k+d} \in (x_1^{k+d+1}, x_2^{k+d+1})$ and, hence, $H^2_{(x_1,x_2)}(A_p^{(2,k,d)})$ is 0. Throughout the rest of this section, we fix the notation, for all $N \ge 1$,

$$I_N := (x_1^N, \ldots, x_n^N) A_p^{(n,k,d)}.$$

Lemma 5.1.18. (a) *For all $N \geq 0$ and all $0 \leq t \leq N$, $s_2^t x_1^{N-t} x_2^N I_{k+1} \subseteq I_{k+N+1}$.*

(b) *For all $N \geq 1$ and $1 \leq t \leq N$,*

$$s_2^{t-1} x_1^{N-t+1} x_2^N (s_2 w^{(t-1)}) - s_2^t x_1^{N-t} x_2^N (s_2 w^{(t)}) \in I_{k+N+1}.$$

(c) *For all $N \geq 1$, $(x_1 x_2)^N (s_2 w^{(0)}) - (x_2 s_2)^N (s_2 w^{(N)}) \in I_{k+N+1}$.*

Proof. (1) Clearly $(s_2^t x_1^{N-t} x_2^N) x_2^{k+1} \in I_{k+N+1}$, and, using $s_1 x_1 + s_2 x_2 = 0$,

$$(s_2^t x_1^{N-t} x_2^N) x_1^{k+1} = x_1^{k+N+1-t} x_2^{N-t} (s_2 x_2)^t = \pm (x_1^{k+N+1} x_2^{N-t} s_1^t) \in I_{k+N+1}.$$

(2) As $s_2^{t-1} x_1^{N-t+1} x_2^N (s_2 w^{(t-1)}) - s_2^t x_1^{N-t} x_2^N (s_2 w^{(t)}) = s_2^t x_1^{N-t} x_2^N (x_1 w^{(t-1)} - s_2 w^{(t)})$, and $x_1 w^{(t-1)} - s_2 w^{(t)} \in I_{k+1}$ by (5.1.17), (a) gives the result.

(3) This part follows from repeated applications of (b).

□

Proposition 5.1.19. *For any p, k, d, $(x_1 x_2)^{k+d} \in I_{k+d+1}$, and as a consequence, $H^2_{(x_1,x_2)}(A_p^{(2,k,d)}) = 0$.*

Proof. Since $A_p^{(n,k,d+1)} \to A_p^{(n,k,d)}$, for all p, n, k, d, we may assume that $d \geq 1$. By construction of $A_p^{(2,k,d)}$ in (5.1.17) and Lemma 5.1.18,

$$(x_1 x_2)^{k+d} \equiv (x_1 x_2)^d (s_2 w^{(0)}) \equiv (x_2 s_2)^d (s_2 w^{(d)}) = 0,$$

where congruences are modulo I_{k+d+1}. □

We have now proven the following result by way of the last proposition, Proposition 5.1.16, and [Ho3, Corollary 10.6].

Proposition 5.1.20. *If R is a complete local domain of positive characteristic and $\dim R \leq 2$, then an R-algebra S is solid over R if and only if S maps to a big Cohen-Macaulay R-algebra.*

5.1.3 The Families of Algebras $A_p^{(n,k,0)}$, $A_p^{(n,k,1)}$, and $A_p^{(n,0,d)}$

The new cases where we can show that $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,k,d)}) = 0$ are when $d \leq 1$ or when $k = 0$. We will present the argument for each case, including two proofs for the $d \leq 1$ case.

Proposition 5.1.21. $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,0,d)}) = 0$, *for all* p, n, d.

Proof. Let congruences be taken modulo $I_2 = (x_1^2, \ldots, x_n^2)A_p^{(n,0,d)}$. Using the defining relations of $A_p^{(n,0,d)}$, as given in (5.1.14),

$$\begin{aligned}
(x_1 \cdots x_n) &\equiv (x_1 \cdots x_n)(s_n w^{(0)}) = (x_1 \cdots x_{n-1})(x_n s_n) w^{(0)} \\
&= -(x_1 \cdots x_{n-1})(x_1 s_1 + \cdots + x_{n-1}s_{n-1}) w^{(0)} \\
&\equiv 0.
\end{aligned}$$

$\square$

We now handle the case of $d \leq 1$. We only need to study $A_p^{(n,k,1)}$ as this case will also take care of the $d = 0$ case since there is a map $A_p^{(n,k,1)} \to A_p^{(n,k,0)}$. Now, let $\epsilon_\ell := (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the ℓ^{th} position. Then

$$A_p^{(n,k,1)} = \frac{\mathbb{Z}/p\mathbb{Z}[x_j, s_j, f_j^{(t_1,\ldots,t_{n-1})}, w^{(0,\ldots,0)} \mid 1 \leq j \leq n, t_1 + \cdots + t_{n-1} \leq 1]}{\begin{pmatrix} x_1 s_1 + \cdots + x_n s_n \\ (x_1 \cdots x_n)^k - x_1^{k+1} f_1^{(0,\ldots,0)} - \cdots - x_n^{k+1} f_n^{(0,\ldots,0)} - s_n w^{(0,\ldots,0)} \\ x_\ell w^{(0,\ldots,0)} - x_1^{k+1} f_1^{\epsilon_\ell} - \cdots - x_n^{k+1} f_n^{\epsilon_\ell} \quad \forall 1 \leq \ell \leq n-1 \end{pmatrix}}.$$

Proposition 5.1.22. *For all* p, n, k, $H^n_{(x_1,\ldots,x_n)}(A_p^{(n,k,1)}) = 0$. *(The first proof will show explicitly that* $(x_1 \cdots x_n)^{kp+p-1} \in I_{kp+p} = (x_1^{kp+p}, \ldots, x_n^{kp+p})A_p^{(n,k,1)}$*.)*

Proof 1. The defining relations of $A_p^{(n,k,1)}$ imply that

$$(x_1 \cdots x_n)^{kp} - (s_n w^{(0,\ldots,0)})^p \in I_{kp+p}$$

and that for all $1 \leq \ell \leq n-1$, each $(x_\ell w^{(0,\ldots,0)})^p$ is in I_{kp+p}. We then have

$$\begin{aligned}(x_1 \cdots x_n)^{p-1}(x_1 \cdots x_n)^{kp} &\equiv (x_1 \cdots x_n)^{p-1}(s_n w^{(0,\ldots,0)})^p \\ &= \pm s_n^{p-1} x_n^{p-2}[s_1(x_2 \cdots x_n)^{p-1}(x_1 w^{(0,\ldots,0)})^p \\ &\quad + \cdots + s_{n-1}(x_1 \cdots x_{n-2})^{p-1}(x_{n-1} w^{(0,\ldots,0)})^p] \\ &\equiv 0\end{aligned}$$

modulo I_{kp+p}, using $x_1 s_1 + \cdots + x_n s_n = 0$. □

Proof 2. The defining relations of $A = A_p^{(n,k,1)}$ give us the following equation

$$\begin{aligned}(x_1 \cdots x_n)^k &= x_1^{k+1}(f_1^{(0,\ldots,0)} + f_1^{\epsilon_1} U_1 + \cdots + f_1^{\epsilon_{n-1}} U_{n-1}) \\ &\quad + \cdots + x_n^{k+1}(f_n^{(0,\ldots,0)} + f_n^{\epsilon_1} U_1 + \cdots + f_n^{\epsilon_{n-1}} U_{n-1}) \\ &\quad + w^{(0,\ldots,0)}(s_n - x_1 U_1 - \cdots - x_{n-1} U_{n-1})\end{aligned}$$

in $A[U_1, \ldots, U_{n-1}]$. By taking p^{th} powers, we also obtain

$$\begin{aligned}(x_1 \cdots x_n)^{kp} &= x_1^{kp+p}((f_1^{(0,\ldots,0)})^p + (f_1^{\epsilon_1})^p U_1^p + \cdots + (f_1^{\epsilon_{n-1}})^p U_{n-1}^p) \\ &\quad + \cdots + x_n^{kp+p}((f_n^{(0,\ldots,0)})^p + (f_n^{\epsilon_1})^p U_1^p + \cdots + (f_n^{\epsilon_{n-1}})^p U_{n-1}^p) \\ &\quad + (w^{(0,\ldots,0)})^p(s_n^p - x_1^p U_1^p - \cdots - x_{n-1}^p U_{n-1}^p)\end{aligned}$$

in $A[U_1^p, \ldots, U_{n-1}^p]$. We can then map $A[U_1^p, \ldots, U_{n-1}^p] \to A_{x_1 \cdots x_n}$ by sending

$$U_j^p \mapsto \frac{-s_j s_n^{p-1}}{x_j^{p-1} x_n}.$$

Since

$$\begin{aligned}s_n^p - x_1^p U_1^p - \cdots - x_{n-1}^p U_{n-1}^p &\mapsto s_n^p + x_1^p \left(\tfrac{s_1 s_n^{p-1}}{x_1^{p-1} x_n}\right) + \cdots + x_{n-1}^p \left(\tfrac{s_{n-1} s_n^{p-1}}{x_{n-1}^{p-1} x_n}\right) \\ &= s_n^p + \tfrac{s_n^{p-1}}{x_n}(x_1 s_1 + \cdots + x_{n-1} s_{n-1}) \\ &= s_n^p + \tfrac{s_n^{p-1}}{x_n}(-x_n s_n) = 0,\end{aligned}$$

and since $(x_1 \cdots x_n)^{p-1}$ clears the denominator of the image of each U_j^p, we obtain

$$(x_1 \cdots x_n)^{kp+p-1} \in I_{kp+p} + \ker(A \to A_{x_1 \cdots x_n}).$$

To finish, note any element of $\ker(A \to A_{x_1 \cdots x_n})$ is killed by $(x_1 \cdots x_n)^N$, for some N. Hence, $(x_1 \cdots x_n)^{N+kp+p-1} \in I_{N+kp+p}$. □

To summarize the state of our knowledge about Questions 5.1.1 and 5.1.2:

Proposition 5.1.23. *If R is a complete local domain of dimension n and characteristic $p > 0$, S is solid, and T is a hollow modification of type (k, d), then $n \geq 3$, $k \geq 1$, and $d \geq 2$.*

5.1.4 The Family of Algebras $A_p^{(3,k,2)}$

We briefly examine one more case, where $\dim R = 3$ and $d = 2$. After changing notation, we have

$$A_p^{(3,k,2)} = \frac{\mathbb{Z}/p\mathbb{Z}[x, y, z, a, b, c, f_j, g_j, h_j, w_0, w_1, w_2 \,|\, 0 \leq j \leq 5]}{\begin{pmatrix} ax + by + cz \\ (xyz)^k - x^{k+1}f_0 - y^{k+1}g_0 - z^{k+1}h_0 - cw_0 \\ xw_0 - x^{k+1}f_1 - y^{k+1}g_1 - z^{k+1}h_1 - cw_1 \\ yw_0 - x^{k+1}f_2 - y^{k+1}g_2 - z^{k+1}h_2 - cw_2 \\ xw_1 - x^{k+1}f_3 - y^{k+1}g_3 - z^{k+1}h_3 \\ yw_2 - x^{k+1}f_4 - y^{k+1}g_4 - z^{k+1}h_4 \\ xw_2 + yw_1 - x^{k+1}f_5 - y^{k+1}g_5 - z^{k+1}h_5 \end{pmatrix}}.$$

At this time, we do not have enough information about $A_p^{(3,k,2)}$ to determine whether or not $H^3_{(x,y,z)}(A_p^{(3,k,2)}) = 0$. Attempts have been made with *Macaulay 2* [M2] to test whether or not certain powers $(xyz)^N$ are contained in $(x^N, y^N, z^N)A_p^{(3,k,2)}$. Even for small values of k and p, the computations have been too complex for *Macaulay 2* to perform successfully. For example, the case $k = 1$ and $p = 2$ was entered into *Macaulay 2*. After twelve days, the program terminated without an answer or even an error message.

For the time being, we will have to leave the question of whether $H^3_{(x,y,z)}(A_p^{(3,k,2)})$ is zero or not as an open problem.

5.2 Solid Algebras and Phantom Extensions

In [HH5], Hochster and Huneke defined the notion of a *phantom extension* for a map $N \to M$ of finitely generated R-modules, where R is a Noetherian ring of characteristic $p > 0$. In this section, we will extend this definition to all R-modules and introduce the concept of a *phantom algebra* over R. For a Noetherian domain R we will show that the solid R-algebras are exactly the phantom algebras. We will also introduce *lim-phantom extensions*, where $\alpha : N \to M$ is lim-phantom if $N \to M'$ is phantom for all finitely generated submodules $M' \subseteq M$ such that $\alpha(N) \subseteq M'$. The primary result, Theorem 5.2.13, will be that if R is a complete local domain, then S is solid if and only if S is phantom if and only if S is lim-phantom. We hope that this perspective on solid R-algebras will help to show that an algebra modification of a solid algebra is still a solid algebra, which holds if and only if the solid algebras are the same as the algebras that map to a big Cohen-Macaulay algebra. (See Lemma 5.1.3.)

Let R be a Noetherian ring of characteristic $p > 0$ throughout this section.

5.2.1 Phantom Extensions

We start with a generalized definition of *phantom extensions* for maps of R-modules that is essentially the definition given by Hochster and Huneke in [HH5] with the requirement that the modules be finitely generated removed.

Definition 5.2.1. A map $\alpha : N \to M$ of arbitrary R-modules is a *phantom extension* if there exists $c \in R^\circ$ such that for all $e \gg 0$, there exists a map $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$. We will also say that M is a *phantom extension* of N. If S is an R-algebra and the structure map $R \to S$ is a phantom extension, we will call S a *phantom R-algebra*.

Remark 5.2.2. In [HH5, Remark 5.4], the following alternate description of a phantom extension is given in the case that R is reduced. When R is reduced, the algebra map $F^e : R \to R$ may be identified with the inclusion map $R \to R^{1/q}$, where $q = p^e$. Thus, $\alpha : N \to M$ is a phantom extension if there exists $c \in R^\circ$ such that for all $e \gg 0$, there exists an $R^{1/q}$-linear map $\gamma_e : R^{1/q} \otimes M \to R^{1/q} \otimes N$ such that $\gamma_e \circ (R^{1/q} \otimes \alpha) = c^{1/q}(\mathrm{id}_{R^{1/q} \otimes N})$.

We will often make use of this remark in the following results. We also note that when R is reduced and M is a phantom extension of R, there exists a map $\gamma_e : R^{1/q} \otimes M \to R^{1/q}$, as in the remark above, for all $e \geq 0$.

Lemma 5.2.3. *Let R be reduced, and let M be an arbitrary R-module. If $\alpha : R \to M$ is a phantom extension, then there exists $c \in R^\circ$ such that for all $e \geq 0$, there is a map $\gamma_e : R^{1/q} \otimes M \to R^{1/q}$ such that $\gamma_e \circ (R^{1/q} \otimes \alpha) = c^{1/q}(\mathrm{id}_{R^{1/q}})$.*

Proof. Since $R \to M$ is phantom, there exists $d \in R^\circ$ such that for all $e \geq e_0$, there is a map $\beta_e : R^{1/q} \otimes M \to R^{1/q}$ such that $\beta_e \circ \alpha_e(r^{1/q}) = d^{1/q} r^{1/q}$, where $\alpha_e(r^{1/q}) = r^{1/q}(1 \otimes \alpha(1))$. Let $S = (R^\circ)^{-1}R$, a finite product of fields since R is reduced. Then $S^{1/q} = ((R^\circ)^{1/q})^{-1}R^{1/q}$ is also a finite product of fields. Thus, the existence of β_e shows that the map $S^{1/q} \to S^{1/q} \otimes M$, induced by α_e, splits for all $e \geq e_0$. Since $S \to S^{1/q}$ is faithfully flat, the map $S \to S \otimes M$, induced by α, also splits.

This implies that there exists $c' \in R^\circ$ and an R-module map $\beta : R \otimes M \to R$ such that $\beta \circ \alpha = c'(\mathrm{id}_R)$. Let $q_0 = p^{e_0}$, and let $c = (c')^{q_0} d \in R^\circ$. For all $e \geq e_0$, define $\gamma_e : R^{1/q} \otimes M \to R^{1/q}$ by $\gamma_e(u) = (c')^{q_0/q} \beta_e(u)$. Then

$$\gamma_e \circ \alpha_e(1) = (c')^{q_0/q} d^{1/q} = c^{1/q}.$$

For $e < e_0$, let $\gamma_e : R^{1/q} \otimes M \to R^{1/q}$ be defined by

$$\gamma_e(u) = (c')^{(q_0/q)-1} d^{1/q} (R^{1/q} \otimes \beta)(u).$$

Then $\gamma_e \circ \alpha_e(1) = (c')^{(q_0/q)-1} d^{1/q}(\beta(\alpha(1))) = (c')^{(q_0/q)-1} d^{1/q} c' = c^{1/q}$. $\square$

Therefore, when R is reduced, if $\alpha : R \to M$ is a phantom extension, then the $e = 0$ condition in the lemma shows that there exists $c \neq 0$ in R and an R-module map $\gamma : M \to R$ such that $\gamma \circ \alpha(1) = c$. We therefore have the following corollary.

Corollary 5.2.4. *Let R be a domain, and let M be any R-module. If M is a phantom extension of R, then M is solid.*

It will also be helpful to show that when R is reduced and M is a phantom extension of R, the map $R \to M$ is always injective.

Lemma 5.2.5. *Let R be reduced, and let M be any R-module. If $\alpha : R \to M$ is a phantom extension, then α is injective.*

Proof. Suppose $\alpha(u) = 0$. Then for all $e \geq 0$, $\mathbf{F}^e(\alpha)(u) = 0$ in $\mathbf{F}^e(M)$ too. For $e \gg 0$, there exists $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(R)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(R)})$ since α is phantom. Therefore, $0 = \gamma_e \circ \mathbf{F}^e(\alpha)(u) = cu^q$, for $e \gg 0$, and so $u \in 0^*_R = 0$ since R is reduced (see Proposition 2.2.3(k)). $\square$

We will now show that when R is a domain the solid R-algebras are exactly the phantom R-algebras by taking advantage of the multiplication in S and the q^{th} powers of elements of S.

Proposition 5.2.6. *Let R be a domain. An R-algebra S is solid if and only if it is a phantom R-algebra.*

Proof. By the previous corollary, we need only show that a solid algebra S is a phantom extension of R. Since S is solid, there exists an R-linear map $\gamma : S \to R$ such that $\gamma(1) = c \neq 0$ in R by Proposition 2.5.2(b). Since γ is R-linear, for all $q \geq 1$, we can define an R-bilinear map $\beta_q : R^{1/q} \times S \to R^{1/q}$ by $\beta_q(r^{1/q}, s) = r^{1/q}\gamma(s^q)^{1/q}$, where $r \in R$. Therefore, β_q induces the $R^{1/q}$-linear map $\gamma_q : R^{1/q} \otimes_R S \to R^{1/q}$ that sends $r^{1/q} \otimes s$ to $r^{1/q}\gamma(s^q)^{1/q}$. Moreover,

$$\gamma_q(r^{1/q} \otimes 1) = r^{1/q}\gamma(1)^{1/q} = c^{1/q}r^{1/q}.$$

Therefore, S is a phantom extension of R. $\square$

We now show that a solid R-module need not be a phantom extension of R, even if the module is finitely-generated and torsion-free.

Example 5.2.7. Let R be a regular local domain of dimension at least 2. Let I be any ideal minimally generated by at least 2 elements, and let $\alpha : R \hookrightarrow I$ be a fixed injection such that $\alpha(1) \neq 0$. We claim that any such extension α is solid, but not a phantom extension of R. Indeed, the inclusion map $I \subseteq R$ shows that I is solid. Since R is a domain, I is also a rank 1 torsion-free submodule of R, and so R is not a direct summand of I via α. (Since I has torsion-free rank 1, $I/\alpha(R)$ is a torsion-module. So, if α is split, then $I = \alpha(R) \oplus W$, where W is a torsion submodule of I, which is not possible unless $W = 0$, but then $I \cong R$. This contradicts the assumption that I is generated by at least two elements.) By [HH5, Theorem 5.13], a phantom extension $R \hookrightarrow M$ over a weakly F-regular ring is actually a split map. Since a regular ring is weakly F-regular (see Proposition 2.2.3(i)) and since there is no split map $R \hookrightarrow I$, we see that I cannot be a phantom extension of R.

5.2.2 Lim-Phantom Extensions

We now introduce the concept of a *lim-phantom extension* of R-modules and demonstrate some connections between this property, phantom extensions, and solid algebras.

Definition 5.2.8. A map $\alpha : N \to M$ of a finitely generated R-module N to an arbitrary R-module M is a *lim-phantom extension* if the map $N \to M'$ is a phantom extension for all finitely generated submodules $M' \subseteq M$ such that $\alpha(N) \subseteq M'$. If S is an R-algebra, and $R \to S$ is lim-phantom, then we will call S a *lim-phantom R-algebra*.

The next lemma is a generalization of [HH5, Proposition 5.7e] and will be used to connect the concepts of phantom and lim-phantom extensions.

Lemma 5.2.9. *Let N, M, M' be any R-modules with maps $N \xrightarrow{\beta} M' \xrightarrow{\theta} M$. Let $\alpha = \theta \circ \beta$. If α is a phantom extension, then β is also phantom.*

Proof. Since α is phantom, there is a $c \in R^{\circ}$ such that for all $e \gg 0$, there exists an R-linear map $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$. Let $\gamma'_e : \mathbf{F}^e(M') \to \mathbf{F}^e(N)$ be the map $\gamma'_e = \gamma_e \circ \mathbf{F}^e(\theta)$. Then

$$\gamma'_e \circ \mathbf{F}^e(\beta) = \gamma_e \circ \mathbf{F}^e(\theta) \circ \mathbf{F}^e(\beta) = \gamma_e \circ \mathbf{F}^e(\theta \circ \beta) = \gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)}).$$

$\square$

The following is an immediate consequence of the previous lemma.

Lemma 5.2.10. *If $N \to M$ is a phantom extension, where N is a finitely generated R-module and M is not necessarily finitely generated, then it is also a lim-phantom extension.*

It is also clear that when M is a finitely generated R-module, a map $N \to M$ is a phantom extension if and only if it is a lim-phantom extension.

We are about to show that solid and lim-phantom algebras are equivalent notions when R is a complete local domain. First, however, we need the following result.

Proposition 5.2.11. *Let R be a domain that is a module-finite extension ring of A, and let M be any R-module. If $\alpha : R \to M$ is a phantom extension over R, then $A \to M$ is a phantom extension over A.*

Proof. Let $\alpha(1) = u \in M$, and let $\alpha_q = R^{1/q} \otimes_R \alpha$. There exists $c \neq 0$ and

$$\gamma_q : R^{1/q} \otimes_R M \to R^{1/q},$$

an $R^{1/q}$-linear map, such that $\gamma_q(1 \otimes u) = c^{1/q}$ for all $q \geq 1$. Let

$$\psi_q : A^{1/q} \otimes_A M \to R^{1/q} \otimes_R M$$

be the natural $A^{1/q}$-linear map induced by the inclusion $A^{1/q} \hookrightarrow R^{1/q}$. Notice that the map γ_q is also an $A^{1/q}$-linear map.

Since R is a domain and $A \subseteq R$ is a module-finite extension of A, R is a finitely generated torsion-free A-module. So, there exists an h and a map $f : R \hookrightarrow A^h$. Since $f(c) \neq 0$, there exists a projection map $g : A^h \twoheadrightarrow A$ such that the composition $\eta = g \circ f : R \to A$ yields an A-linear map with $\eta(c) = d \neq 0$ in A. We can then define $\eta_q : R^{1/q} \to A^{1/q}$ by $\eta_q(r^{1/q}) = \eta(r)^{1/q}$, where $r \in R$, for all $q \geq 1$.

Now, let $\theta_q : A^{1/q} \otimes_A M \to A^{1/q}$ be given by $\theta_q = \eta_q \circ \gamma_q \circ \psi_q$. Then θ_q is $A^{1/q}$-linear, and

$$\theta_q(1 \otimes u) = \eta_q \circ \gamma_q(1 \otimes u) = \eta_q(c^{1/q}) = \eta(c)^{1/q} = d^{1/q}.$$

Hence, $A \to M$ is a phantom extension over A. $\square$

We now prove the last necessary lemma before our main result. First, recall that an injection of R-modules $N \to M$ is *pure* if $W \otimes N \to W \otimes M$ is an injection, for all R-modules W. When M/N is finitely presented, the map is pure if and only if the map splits, see [HR, Corollary 5.2].

Lemma 5.2.12. *Let (R, m) be a complete local domain of dimension d, and let M be any R-module. If $\alpha : R \to M$ is a lim-phantom extension, then M is solid.*

Proof. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of all finitely generated submodules of M such that $\alpha(R) \subseteq M_\lambda$. Then $R \to M_\lambda$ is phantom for all λ. Since R is a complete local domain, R is a module-finite local extension of a regular local ring (A, n). By Proposition 5.2.11, since each $R \to M_\lambda$ is a phantom extension over R, $A \to M_\lambda$ is a phantom extension over A. Since $A \to M_\lambda$ is phantom, by Lemma 5.2.5, $A \hookrightarrow M_\lambda$ for all λ. As A is regular, and so weakly F-regular, [HH5, Theorem 5.13] shows that all the maps $A \hookrightarrow M_\lambda$ are split and, thus, are pure because M_λ is finitely generated over A. Since $M = \varinjlim_\lambda M_\lambda$ (over R or A) and all maps $A \hookrightarrow M_\lambda$ are pure, [HH7, Lemma 2.1i] tells us that $A \hookrightarrow M$ is pure over A. Hence, $H^d_n(A) \hookrightarrow H^d_n(A) \otimes_A M \cong H^d_n(M)$, and so $H^d_n(M) \neq 0$. Since the radical of nR in R is m, $H^d_m(M) \cong H^d_n(M)$ as R-modules. By the local cohomology criterion, M is solid over R. $\square$

We can finally state our main result of the section.

Theorem 5.2.13. *Let R be a complete local domain, and let S be an R-algebra. Then S is solid if and only if S is phantom if and only if S is lim-phantom.*

Proof. We know that solid implies phantom by Proposition 5.2.6. By Lemma 5.2.10 we see that phantom implies lim-phantom. The final implication follows directly from the previous lemma. $\square$

CHAPTER 6

Seed Algebras

In this chapter we will study the class of algebras over a local ring R that map to a big Cohen-Macaulay algebra over R. We call these algebras *seeds* over R. Seeds are closely related to the class of solid algebras by Hochster's Theorem 2.5.5. and the fact that solid algebras are seeds when the dimension of the base ring is at most 2 (see [Ho3, Theorem 12.5]). Although seeds and solid algebras are not generally the same in equal characteristic 0 (see [Ho3, Example 10.7]), it is still possible that seeds and solid algebras are the same in positive characteristic, and even in mixed characteristic, for complete local domains. This study of seeds leads to new understanding of big Cohen-Macaulay algebras and we hope that it will eventually lead to new insight into solid algebras.

In the following section, we will show how seeds in positive characteristic can be mapped to big Cohen-Macaulay algebras with some special properties. In the second section, we will show that if R is a module-finite extension of a Cohen-Macaulay ring A, and S is an R-algebra, then a colon-killer (see Definition 6.2.1) in S, with respect to parameters from A, has a power that is a colon-killer with respect to parameters from R. As a result, if S is a big Cohen-Macaulay A-algebra, then it is also a big Cohen-Macaulay R-algebra. We will also define a special class of colon-killers, called *durable colon-killers*, whose existence can be used to classify when an algebra maps to a big Cohen-Macaulay algebra in positive characteristic.

In the third section we will define *minimal seeds*, which are seeds that have no proper homomorphic image that is a seed (*cf.* minimal solid algebras, [Ho3, Section 6]). We demonstrate that, without any finiteness conditions, every seed maps to a minimal seed. We will first show that in positive characteristic, minimal seeds are reduced and later show that they are, in fact, domains. These results are parallel to analogous results for minimal solid algebras, as Noetherian solid algebras map onto minimal solid algebras which are domains.

In the fourth section, we will examine module-finite and integral extensions of seeds and show that these extensions are also seeds in positive characteristic. This result may be viewed as a generalization of the fact that every complete local domain maps to a big Cohen-Macaulay algebra. The proof is complicated. One key element is the construction of a durable colon-killer in a particular module-finite extension of a big Cohen-Macaulay algebra.

In the fifth section, we will use the result about integral extensions of seeds in positive characteristic to show that minimal seeds are in fact domains and that seeds can be mapped to quasilocal big Cohen-Macaulay algebras domains that are absolutely integrally closed and m-adically separated.

We finally will use our new insight into seeds to prove some new facts about big Cohen-Macaulay algebras over complete local domains in positive characteristic. We will show that the tensor product of seeds is also a seed and that the property of being a seed is preserved under base change between complete local domains (*cf.* Proposition 2.5.2(a),(e)). As a consequence, we can use the classes of big Cohen-Macaulay algebras over complete local domains to define a closure operation equivalent to tight closure in characteristic $p > 0$.

6.1 Definition and Properties of Seeds

Definition 6.1.1. For any local Noetherian ring R, an R-algebra S is called a *seed* over R if S maps to a big Cohen-Macaulay R-algebra.

Using our new terminology, Proposition 2.3.8 implies that S is a seed if and only if S does not have a bad sequence of algebra modifications. Based on this characterization of seeds, a direct limit of seeds is still a seed.

Lemma 6.1.2. *Let (R, m) be a local Noetherian ring, and let $S = \varinjlim_\lambda S_\lambda$ be a direct limit of a directed set of R-algebras. Then S is a seed if and only if each S_λ is a seed.*

Proof. Since S is an S_λ-algebra for all λ, it is clear that if S is a seed, then so is each S_λ. Conversely, suppose that S is not a seed. We will find an S_λ that also has a bad sequence of modifications.

As S is not a seed, it has a bad sequence of algebra modifications

$$S = S^{(0)} \to S^{(1)} \to \cdots S^{(t)},$$

where $1 = r_1 s_1^{(t)} + \cdots + r_{k_t} s_{k_t}^{(t)}$ in $S^{(t)}$ (with $r_j \in m$),

$$S^{(i+1)} = \frac{S^{(i)}[U_1^{(i)}, \ldots, U_{k_i}^{(i)}]}{s^{(i)} - \sum_{j=1}^{k_i} x_j^{(i)} U_j^{(i)}},$$

$s^{(i)} x_{k_i+1} = \sum_{j=1}^{k_i} x_j^{(i)} s_j^{(i)}$ in $S^{(i)}$, and $x_1^{(i)}, \ldots, x_{k_i+1}^{(i)}$ is part of a system of parameters in R, for all $0 \le i \le t-1$.

It is straightforward, but rather tedious, to show that some S_λ has a bad sequence of modifications induced by the bad sequence originating with S because the sequence is finite and involves only finitely many relevant elements. This S_λ is not a seed. $\square$

We next show that in positive characteristic, we can use the Frobenius endomorphism and its iterates to map any seed to a reduced and perfect big Cohen-Macaulay algebra.

Since we do not want to limit ourselves to the study of only reduced rings, we will generalize our definition of R^∞ to include non-reduced rings. From now on, we will let R^∞ denote the direct limit of the directed system

$$R \to \mathbf{F}(R) \to \mathbf{F}^2(R) \cdots \to \mathbf{F}^e(R) \to \cdots,$$

where $\mathbf{F}^e$ is the iterated Peskine-Szpiro functor. Note that $R^\infty = (R_{\text{red}})^\infty$, where R_{red} is the quotient of R obtained by killing all nilpotents. If R is already reduced, then our new definition of R^∞ is isomorphic to the original definition.

Lemma 6.1.3. *Let R be a local Noetherian ring of positive characteristic p. If B is a big Cohen-Macaulay R-algebra, then there is a homomorphism $B \to C$ such that C is a reduced big Cohen-Macaulay R-algebra with $C = C^\infty$. Moreover, if B is quasilocal, then C is also quasilocal.*

Proof. Let $C = B^\infty$. Let $c \in (x_1, \ldots, x_k)C :_C x_{k+1}$, for some partial system of parameters in R. Since C is a direct limit of the $\mathbf{F}^e(B)$, there exists an $e \in \mathbb{N}$ such that $c \in (x_1, \ldots, x_k)\mathbf{F}^e(B) :_{\mathbf{F}^e(B)} x_{k+1}$. Since $\mathbf{F}^e(B)$ is just B as a ring, we have $c \in (x_1^q, \ldots, x_k^q)B :_B x_{k+1}^q$. Since B is a big Cohen-Macaulay R-algebra, c is in $(x_1^q, \ldots, x_k^q)B$, so that $c \in (x_1, \ldots, x_k)\mathbf{F}^e(B)$. Therefore, $c \in (x_1, \ldots, x_k)C$. Thus, every system of parameters in R is a possibly improper regular sequence on C. If $1 \in (x_1, \ldots, x_k)C$, then $1 \in (x_1, \ldots, x_k)\mathbf{F}^e(B)$, for some e, and so $1 \in (x_1^q, \ldots, x_k^q)B$, a contradiction. Hence, every system of parameters in R is a regular sequence on C, and C is a big Cohen-Macaulay R-algebra.

Suppose that $c^N = 0$ in C. Then there exists an $\mathbf{F}^e(B)$ such that c is the image of $b \in \mathbf{F}^e(B)$ and $b^N = 0$ in $\mathbf{F}^e(B)$. Choose $p^{e'} \geq N$, then b maps to $b^{p^{e'}} = 0$ in $\mathbf{F}^{e+e'}(B)$. Since c is also the image of $b^{p^{e'}}$, $c = 0$ in C. Thus, C is reduced.

Now, let $c \in C$. For a given p^e, we will find $d \in C$ such that $d^{p^e} = c$. Choose e' such that c is the image of $b \in F^{e'}(B)$. Then c is also the image of b^{p^e} in $\mathbf{F}^{e'+e}(B)$. Let d be the image of the copy of b in $\mathbf{F}^{e'+e}(B)$. Then $d^{p^e} = c$ as desired.

Finally, if B is quasilocal, then C is a direct limit of quasilocal rings via local maps. Therefore, C is also quasilocal. □

Although the above result shows that every seed in positive characteristic can be mapped to a reduced big Cohen-Macaulay algebra, we will show later (Proposition 6.5.6) that any reduced seed (in any characteristic) can be modified into a reduced big Cohen-Macaulay algebra.

Lemma 6.1.4. *Let (R, m) be a local Noetherian ring. If B is a big Cohen-Macaulay R-algebra and $\mathfrak{p}$ is any prime ideal of B containing mB, then $B_\mathfrak{p}$ is also a big Cohen-Macaulay R-algebra. Moreover, if B is reduced (resp., R has positive characteristic and B is reduced and perfect), then $B_\mathfrak{p}$ is still reduced (resp., reduced and perfect).*

Proof. Given $x_{k+1}(r/u) \in (x_1, \ldots, x_k)B_\mathfrak{p}$, where $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, we may assume $u = 1$ in showing that the relation is trivial. Therefore, there exists a $v \in B \setminus \mathfrak{p}$ such that $x_{k+1}(rv) \in (x_1, \ldots, x_k)B$. Since B is a big Cohen-Macaulay algebra, $rv \in (x_1, \ldots, x_k)B$, and thus $r/1 \in (x_1, \ldots, x_k)B_\mathfrak{p}$ as needed. Furthermore, since $mB \neq B$ and $\mathfrak{p} \supseteq mB$, we see that $mB_\mathfrak{p} \neq B_\mathfrak{p}$, and so $B_\mathfrak{p}$ is a big Cohen-Macaulay algebra.

The other claims follow from the following easy lemma. □

Lemma 6.1.5. *If S is any reduced ring and U is a multiplicatively closed set in S, then $U^{-1}S$ is also reduced. If in addition, S has positive characteristic and is perfect,* *then $U^{-1}S$ is also perfect.*

Proof. The first claim is well known.

Now suppose that S has positive characteristic and is perfect. Given s/u in $U^{-1}S$, we need to find a q^{th} root of s/u. Since S is perfect, we have $s = a^q$ and $u = b^q$, where a and b are in S. Then ab^{q-1}/u is an element of $U^{-1}S$, and is a q^{th} root of s/u:

$$\left(\frac{ab^{q-1}}{u}\right)^q = \frac{a^q(b^q)^{q-1}}{u^q} = \frac{su^{q-1}}{u^q} = \frac{s}{u}.$$

□

Another operation that we will use is the separated completion of a big Cohen-Macaulay R-algebra with respect to the maximal ideal of R. The result of Bartijn and Strooker, [Bar-Str, Theorem 1.7], shows that the separated completion of a big Cohen-Macaulay algebra is still a big Cohen-Macaulay algebra. (In fact, it shows that if a single system of parameters is a regular sequence on B, then every system of parameters is a regular sequence on the separated completion of B.) What will be especially useful is the fact that this completion operation will give us an m-adically separated big Cohen-Macaulay algebra while preserving the other properties we have worked with earlier. We will start by showing the completion operation preserves whether a ring is reduced and perfect in positive characteristic.

Lemma 6.1.6. *If A is a reduced and perfect ring of positive characteristic p, and I is an ideal of A, then the I-adic completion $\widehat{A}$ of A is reduced and perfect.*

Proof. By definition,

$$\widehat{A} = \{a = (a_1, a_2, a_3, \ldots) \in \prod_j A/I^j \mid a_k \equiv a_j \pmod{I^j}, \quad \forall k > j\}.$$

If $a^n = 0$ in $\widehat{A}$, then there exists q, a power of p, such that $a^q = 0$, so that $a_k^q \in I^k$ for all k. Given any index j, there exists an integer $k(j)$ such that $I^{k(j)} \subseteq (I^j)^{[q]}$.

Therefore, for any j, we can find $k(j) \geq j$ such that $a_{k(j)}^q \in (I^j)^{[q]}$. Since A is perfect, $a_{k(j)} \in I^j$, and since $k(j) \geq j$, we have $a_j \in I^j$. Hence, $a = 0$, and $\widehat{A}$ is reduced.

Given $a = (a_1, a_2, \ldots) \in \widehat{A}$, we will now find an element $b \in \widehat{A}$ such that $b^q = a$. Indeed, let $b = (a_{k(1)}^{1/q}, a_{k(2)}^{1/q}, \ldots)$, where $k(j)$ is chosen so that $k(j) \geq j$, $k(j) \geq k(j-1)$, and $I^{k(j)} \subseteq (I^j)^{[q]}$. If $i \geq k(j)$, then $a_i \equiv a_{k(j)} \pmod{I^{k(j)}}$, so that $a_i \equiv a_{k(j)} \pmod{(I^j)^{[q]}}$. Since A is perfect, we can take q^{th} roots to see that $a_i^{1/q} \equiv a_{k(j)}^{1/q} \pmod{I^j}$, which shows that b is a well-defined element of $\widehat{A}$. Finally, $b^q = (a_{j(1)}, a_{j(2)}, \ldots)$, which is easily seen to be equal to a. □

This lemma is the last piece we need to show that seeds map to big Cohen-Macaulay algebras with certain rather useful properties. In the fifth section, we will show that seeds map to big Cohen-Macaulay algebras with even stronger properties.

Proposition 6.1.7. *Let (R, m) be a Noetherian local ring of positive characteristic. Every seed over R maps to a big Cohen-Macaulay R-algebra B that is reduced, perfect, quasilocal, and m-adically separated.*

Proof. By Lemma 6.1.4, a seed maps to a quasilocal big Cohen-Macaulay R-algebra. By Lemma 6.1.3 it maps further to a quasilocal, reduced, and perfect big Cohen-Macaulay algebra B. By [Bar-Str, Theorem 1.7], the m-adic completion $\widehat{B}$, which is m-adically separated, is still a big Cohen-Macaulay algebra, and the previous lemma shows that it is also reduced and perfect. Finally, since B is quasilocal, it is easy to see that the set of Cauchy sequences in $\widehat{B}$ with each (equivalently, one) element of the sequence not a unit forms a unique maximal ideal of $\widehat{B}$ so that $\widehat{B}$ is still quasilocal. □

6.2 Colon-Killers and Seeds

Hochster and Huneke used colon-killers (also called *Cohen-Macaulay multipliers*) in [HH2] and [HH3] as tools for proving the existence of big Cohen-Macaulay algebras in positive characteristic. Not surprisingly, the existence of such elements in algebras over a local ring will be useful in determining whether an algebra is a seed or not. We shall work with a slightly generalized version of their definition, and will define a special class of colon-killers that will help us determine when an R-algebra is a seed.

Definition 6.2.1. Let R be a local Noetherian ring, S an R-algebra, and M an arbitrary S-module. An element $c \in S$ is a *colon-killer* for M over R if

$$c((x_1, \ldots, x_k)M :_M x_{k+1}) \subseteq (x_1, \ldots, x_k)M,$$

for each partial system of parameters $x_1, \ldots, x_{k+1}$ in R.

We will soon prove that if B is an S-algebra, where S is a local Noetherian integral extension of a Noetherian local ring R, and B is a a big Cohen-Macaulay R-algebra, then B is also a big Cohen-Macaulay S-algebra. We will actually prove a more general statement showing that a colon-killer for an S-module M over R has a power that is a colon-killer for M over S. First, we need the next lemma connecting colon-killers and Koszul homology. (For an introduction to Koszul complexes and Koszul homology, we refer the reader to [BH, Section 1.6].)

Lemma 6.2.2. *Let R be a local Noetherian ring, let S be an R-algebra, and let M be an arbitrary S-module. If $c \in S$ is nonzero, then the following are equivalent:*

(i) *Some power of c is a colon-killer for M over R.*

(ii) *Some power of c kills all Koszul homology modules $H_i(x_1, \ldots, x_k; M)$, for all $i \geq 1$ and all partial systems of parameters $x_1, \ldots, x_k$.*

(iii) *Some power of c kills $H_1(x_1, \dots, x_k; M)$, for all partial systems of parameters $x_1, \dots, x_k$.*

Proof. (ii) $\Rightarrow$ (iii) is obvious. For (iii) $\Rightarrow$ (i), let $\mathbf{x} = x_1, \dots, x_k$ be part of a system of parameters for R, and let $\mathbf{x}' = x_1, \dots, x_{k-1}$. We obtain a short exact sequence

$$0 \to \frac{H_i(\mathbf{x}'; M)}{x_k H_i(\mathbf{x}'; M)} \to H_i(\mathbf{x}; M) \to \mathrm{Ann}_{H_{i-1}(\mathbf{x}'; M)} x_k \to 0 \tag{6.2.3}$$

for all i, from [BH, Corollary 1.6.13(a)]. In the case $i = 1$, we see that there is a surjection of $H_1(\mathbf{x}; M)$ onto the module $((\mathbf{x}')M :_M x_k)/(\mathbf{x}')M$, which implies that the latter module is killed by the same power of c that kills the former.

For (i) $\Rightarrow$ (ii), assume without loss of generality that c itself is a colon-killer for M over R. We will use induction on k to show that $c^{2^{k-1}}$ kills $H_i(x_1, \dots, x_k; M)$, for $i \geq 1$. If $k = 1$, then $H_1(x_1; M)$ is the only nonzero Koszul homology module, and it is isomorphic to $\mathrm{Ann}_M x_1 = (0 :_M x_1)$. Since c is a colon-killer for M, c kills $H_1(x_1; M)$. Now let $k \geq 2$, $\mathbf{x} = x_1, \dots, x_k$, $\mathbf{x}' = x_1, \dots, x_{k-1}$, and suppose that $c^{2^{k-2}}$ kills $H_i(\mathbf{x}'; M)$, for $i \geq 1$. Using the sequence (6.2.3), we see that $c^{2^{k-1}}$ kills $H_i(\mathbf{x}; M)$, for all $i \geq 2$, by the inductive hypothesis, and $c^{2^{k-2}+1}$ kills $H_1(\mathbf{x}; M)$ by the inductive hypothesis together with c being a colon-killer. Therefore, if $N = 2^{\dim R - 1}$, then c^N kills all of the relevant Koszul homology modules. $\square$

From this lemma, we can obtain our result on colon-killers.

Proposition 6.2.4. *Let S be a Noetherian local ring that is an integral extension of a local Noetherian ring R. Let M be an arbitrary S-module. If $c \in R$ kills all Koszul homology modules $H_i(x_1, \dots, x_k; M)$, for all $i \geq 1$, and all partial systems of parameters $x_1, \dots, x_k$ in R, then c^N kills $H_i(y_1, \dots, y_k; M)$, for all $i \geq 1$, and all partial systems of parameters $y_1, \dots, y_k$ in S, for some N. Consequently, if c is a colon-killer for M over R, then a power of c is a colon-killer for M over S.*

Proof. Based on the previous lemma, it is enough to prove the first claim. Let $\mathbf{y} = y_1, \ldots, y_k$ be part of a system of parameters for S. Since $R \hookrightarrow S$ is integral, $R/((\mathbf{y}) \cap R) \hookrightarrow S/(\mathbf{y})$ is also integral, so that

$$\dim R/((\mathbf{y}) \cap R) = \dim S/(\mathbf{y}) = \dim S - k = \dim R - k.$$

We claim that $(\mathbf{y}) \cap R$ contains a partial system of parameters $\mathbf{x} = x_1, \ldots, x_k$ for R. Indeed, we proceed by induction on k, where the case $k = 0$ is trivial. We can then assume without loss of generality that $k = 1$ and obtain our result from the general fact that if I is an ideal of R such that $\dim R/I < \dim R$, then I contains a parameter of R. If not, then for every $x \in I$, there exists a prime ideal $\mathfrak{p}$ of R such that $x \in \mathfrak{p}$ and $\dim R/\mathfrak{p} = \dim R$. Therefore, I is contained in the union of such prime ideals, and so by prime avoidance, I is contained in a prime $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$. We have a contradiction, which implies that I contains a parameter.

Thus, there exists part of a system of parameters $\mathbf{x} = x_1, \ldots, x_k$ of R such that $(\mathbf{x})S \subseteq (\mathbf{y})$. Then a power of c kills $H_i(\mathbf{y}; M)$, for all $i \geq 1$ by the following lemma, where the power depends only on k, which in turn is bounded by $\dim R$. $\square$

Lemma 6.2.5. *Let S be any ring and M any S-module. Suppose $(x_1, \ldots, x_k)S \subseteq (y_1, \ldots, y_k)S = (\mathbf{y})S$ and that $c \in S$ kills $H_i(x_1, \ldots, x_m; M)$, for all $i \geq 1$ and all $1 \leq m \leq k$. Then $c^{D(m)}$ kills $H_{k+1-m}(\mathbf{y}; M)$, for $1 \leq m \leq k$, where $D(1) = 1$, and $D(m) = 2^{k-2}D(m-1) + 2$, for $m \geq 2$.*

Proof. We will use induction on m. For $m = 1$, we need c to kill $H_k(\mathbf{y}; M)$, but

$$H_k(\mathbf{y}; M) \cong \operatorname{Ann}_M(\mathbf{y}) \subseteq \operatorname{Ann}_M(x_1) \cong H_1(x_1; M),$$

which implies what we want.

Now, let $m \geq 2$. The hypothesis on c together with Lemma 6.2.2 implies that c is a colon-killer for M with respect to subsequences of $x_1, \ldots, x_k$, but this implies that

c is a colon-killer for M/x_1M with respect to subsequences of $x_2, \ldots, x_k$. The proof of Lemma 6.2.2 then implies that $c^{2^{k-2}}$ kills $H_i(x_2, \ldots, x_m; M/x_1M)$, for all $i \geq 1$ and all $2 \leq m \leq k$.

For the induction, suppose that $c^{D(m-1)}$ kills $H_{k+2-m}(\mathbf{y}; M)$. Consider the exact sequence

$$0 \to \mathrm{Ann}_M x_1 \to M \xrightarrow{x_1} M \to M/x_1M \to 0,$$

from which we obtain two short exact sequences

$$0 \to \mathrm{Ann}_M x_1 \to M \to x_1M \to 0$$
$$0 \to x_1M \to M \to M/x_1M \to 0.$$

These sequences then induce long exact sequences in Koszul homology:

$$H_{i+1}(\mathbf{y}; M) \xrightarrow{f_{i+1}} H_{i+1}(\mathbf{y}; x_1M) \to H_i(\mathbf{y}; \mathrm{Ann}_M x_1) \to H_i(\mathbf{y}; M) \xrightarrow{f_i} H_i(\mathbf{y}; x_1M)$$
$$H_{i+1}(\mathbf{y}; x_1M) \xrightarrow{g_{i+1}} H_{i+1}(\mathbf{y}; M) \to H_{i+1}(\mathbf{y}; M/x_1M) \to H_i(\mathbf{y}; x_1M) \xrightarrow{g_i} H_i(\mathbf{y}; M),$$

which in turn yield short exact sequences, for all $i \geq 0$,

$$(*_i) \qquad 0 \to \frac{H_{i+1}(\mathbf{y}; x_1M)}{f_{i+1}(H_{i+1}(\mathbf{y}; M))} \to H_i(\mathbf{y}; \mathrm{Ann}_M x_1) \to \ker(f_i) \to 0$$

$$(\#_i) \qquad 0 \to \frac{H_{i+1}(\mathbf{y}; M)}{g_{i+1}(H_{i+1}(\mathbf{y}; x_1M))} \to H_{i+1}(\mathbf{y}; M/x_1M) \to \ker(g_i) \to 0.$$

Since the map of homology

$$H_i(\mathbf{y}; M) \xrightarrow{f_i} H_i(\mathbf{y}; x_1M) \xrightarrow{g_i} H_i(\mathbf{y}; M)$$

is induced by the composition $M \to x_1M \to M$, which is multiplication by x_1, and since $x_1 \in (\mathbf{y})S$, $g_i \circ f_i = 0$, for all $i \geq 0$.

Since $c\mathrm{Ann}_M x_1 = 0$ by hypothesis, $(*_i)$ implies that

$$cH_{i+1}(\mathbf{y}; x_1M) \subseteq f_{i+1}(H_{i+1}(\mathbf{y}; M)) \subseteq \ker(g_{i+1}),$$

for all $i \geq 0$. Applying the inductive hypothesis to M/x_1M implies that $c^{2^{k-2}D(m-1)}$ kills $H_{k+2-m}(\mathbf{y}; M/x_1M)$. Then $(\#_{k+1-m})$ shows that $c^{2^{k-2}D(m-1)}\ker(g_{k+1-m}) = 0$. Therefore, $c^{2^{k-2}D(m-1)+1}$ kills $H_{k+1-m}(\mathbf{y}; x_1M)$. To finish, notice that $c^{2^{k-2}D(m-1)+1} = c^{D(m)-1}$, which kills the image of $H_{k+1-m}(\mathbf{y}; M)$ inside $H_{k+1-m}(\mathbf{y}; x_1M)$ under the map f_{k+1-m}. Thus, $c^{D(m)-1}H_{k+1-m}(\mathbf{y}; M)$ is contained in $\ker(f_{k+1-m})$, which is killed by c, using $(*_{k+1-m})$, and so $c^{D(m)}$ kills $H_{k+1-m}(\mathbf{y}; M)$, as needed. □

Since 1 is a colon-killer for a big Cohen-Macaulay algebra, the previous result gives us the promised result about big Cohen-Macaulay modules (and algebras).

Corollary 6.2.6. *Let S be a Noetherian local ring that is also an integral extension of a local Noetherian ring R. If M is an S-module and a big Cohen-Macaulay R-module, then M is a big Cohen-Macaulay S-module.*

We will now introduce another notion of a colon-killer that will be very useful for us in the following sections when we need to determine whether a ring is a seed.

Definition 6.2.7. For a local Noetherian ring (R, m) and an R-algebra S, an element $c \in S$ is called a *weak durable colon-killer* over R if for some system of parameters $x_1, \ldots, x_n$ of R,

$$c((x_1^t, \ldots, x_k^t)S :_S x_{k+1}^t) \subseteq (x_1^t, \ldots, x_k^t)S,$$

for all $1 \leq k \leq n-1$ and all $t \in \mathbb{N}$, and if for any $N \geq 1$, there exists $k \geq 1$ such that $c^N \notin m^kS$. An element $c \in S$ will be simply called a *durable colon-killer* over R if it is a weak durable colon-killer for every system of parameters of R.

Notice that if $S = R$, then all colon-killers in R that are not nilpotent are durable colon-killers. So, if R is a domain, or even reduced, all colon-killers are durable colon-killers. Also, if B is a big Cohen-Macaulay algebra over R, then 1 is a durable colon-killer. We can now use the existence of durable colon-killers to characterize when an algebra is a seed by adapting the proof of [Ho3, Theorem 11.1].

Theorem 6.2.8. *Let (R, m) be a local Noetherian ring of positive characteristic p, and let S be an R-algebra. Then S is a seed if and only if there is a map $S \to T$ such that T has a durable colon-killer c if and only if there is a map $S \to T$ such that T has a weak durable colon-killer c.*

Proof. If S is a seed, then $S \to B$, for some big Cohen-Macaulay R-algebra B. As pointed out above, 1 is a durable colon-killer in B, so $T = B$ will suffice. For the converses, we will modify the proof of [Ho3, Theorem 11.1] to obtain our result. We will show that the existence of a (weak) durable colon-killer in an S-algebra T implies that S is a seed. (All parenthetical remarks will apply to the case that T only possesses a weak durable colon-killer.)

Suppose that $S \to T$ such that T has a (weak) durable colon-killer c (with respect to a fixed system of parameters in R). Let $S^{(0)} := S$, and given $S^{(i)}$ for $0 \le i \le t-1$, let

$$S^{(i+1)} := \frac{S^{(i)}[U_1^{(i)}, \ldots, U_{k_i}^{(i)}]}{s^{(i)} - \sum_{j=1}^{k_i} x_j^{(i)} U_j^{(i)}},$$

where $x_1^{(i)}, \cdots, x_{k_i+1}^{(i)}$ is a system of parameters for R (the fixed system of parameters in the latter case), and $x_{k_i+1}^{(i)} s^{(i)} = \sum_{j=1}^{k_i} x_j^{(i)} s_j^{(i)}$ is a relation in $S^{(i)}$. Then

$$S = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(t)}$$

is a finite sequence of algebra modifications. Suppose to the contrary that S is not a seed and the sequence is bad, so that $1 \in mS^{(t)}$. (In the weak case, we are supposing that S does not map to an S-algebra where the fixed system of parameters is a regular sequence.) We can then write

$$1 = \sum_{j=1}^{n} r_j w_j, \tag{6.2.9}$$

where $r_j \in m$ and $w_j \in S^{(t)}$, for all j.

We will construct inductively homomorphisms $\psi_e^{(i)}$ from each $S^{(i)}$ to $\mathbf{F}^e(T)$ forming a commutative diagram:

$$\begin{tikzcd}
\mathbf{F}^e(T)_c \arrow[r, equal] & \mathbf{F}^e(T)_c \arrow[r, equal] & \mathbf{F}^e(T)_c \arrow[r, equal] & \cdots \arrow[r, equal] & \mathbf{F}^e(T)_c \\
S^{(0)} \arrow[u, "\psi_e^{(0)}"] \arrow[r] & S^{(1)} \arrow[u, "\psi_e^{(1)}"] \arrow[r] & S^{(2)} \arrow[u, "\psi_e^{(2)}"] \arrow[r] & \cdots \arrow[r] & S^{(t)}. \arrow[u, "\psi_e^{(t)}"]
\end{tikzcd}$$

In order to construct the maps we need to keep track of bounds, independent of $q = p^e$, associated with the images of certain elements of each $S^{(i)}$. For all $1 \le i \le t$, we will use reverse induction to define a finite subset Γ_i of $S^{(i)}$ and positive integers $b(i)$. We will then inductively define positive integers $\beta(i)$ and $B(i)$, which will be the necessary bounds.

First, let

$$\Gamma_t := \{w_1, \ldots, w_n\},$$

where the w_j are from relation (6.2.9). Now, given Γ_{i+1} (with $0 \le i \le t-1$), each element can be written as a polynomial in the $U_j^{(i)}$ with coefficients in $S^{(i)}$. Let $b(i+1)$ be the largest degree of any such polynomial. For $i \ge 1$, let Γ_i be the set of all coefficients of these polynomials together with $s^{(i)}, s_1^{(i)}, \cdots, s_{k_i}^{(i)}$. Now define $\beta(1) := 1$, $B(1) := b(1)$, and given $B(i)$ for $1 \le i \le t-1$, let

$$\beta(i+1) := B(i) + 1 \qquad \text{and} \qquad B(i+1) := \beta(i+1)b(i+1) + B(i).$$

Notice that, as claimed, all $\beta(i)$ and $B(i)$ are independent of q.

Fix $q = p^e$. By hypothesis, we have a map $S^{(0)} = S \to T$ that can be naturally extended to a map $\psi_e^{(0)} : S^{(0)} \to \mathbf{F}^e(T)_c$ by composing with $T \to \mathbf{F}^e(T)_c$. We next define a map $\psi_e^{(1)} : S^{(1)} \to \mathbf{F}^e(T)_c$ that extends $\psi_e^{(0)}$, maps the $U_j^{(0)}$ to the cyclic $\mathbf{F}^e(T)$-submodule $c^{-1}\mathbf{F}^e(T) = c^{-\beta(1)}\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$, and maps Γ_1 to $c^{-b(1)}\mathbf{F}^e(T) = c^{-B(1)}\mathbf{F}^e(T)$ inside $\mathbf{F}^e(T)_c$. To do this we need only find appropriate images of the $U_j^{(0)}$ such that the image of $s^{(0)} - \sum_{j=1}^{k_0} x_j^{(0)} U_j^{(0)}$ maps to 0.

Since

$$x_{k_0+1}^{(0)} s^{(0)} = \sum_{j=1}^{k_0} x_j^{(0)} s_j^{(0)},$$

we have

$$(x_{k_0+1}^{(0)})^q \psi_e^{(0)}(s^{(0)}) = \sum_{j=1}^{k_0} (x_j^{(0)})^q \psi_e^{(0)}(s_j^{(0)})$$

in $\mathbf{F}^e(T)_c$, where the the image of $\psi_e^{(0)}$ is contained in the image of $\mathbf{F}^e(T)$ inside $\mathbf{F}^e(T)_c$. As c is a (weak) durable colon-killer in T, and so also a (weak) colon-killer in $\mathbf{F}^e(T)$,

$$c\psi_e^{(0)}(s^{(0)}) = \sum_{j=1}^{k_0} (x_j^{(0)})^q \sigma_j^{(0)},$$

where the $\sigma_j^{(0)}$ are in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. If we define $\psi_e^{(1)}$ such that $U_j^{(0)} \mapsto c^{-1}\sigma_j^{(0)}$, then we have accomplished our goal for $\psi_e^{(1)}$ because the elements of Γ_1 can be written as polynomials in the $U_j^{(0)}$ of degree at most $b(1) = B(1)$ with coefficients in S.

Now suppose that for some $1 \le i \le t-1$ we have a map $\psi_e^{(i)} : S^{(i)} \to \mathbf{F}^e(T)_c$, where the $U_j^{(i-1)}$ all map to $c^{-\beta(i)}\mathbf{F}^e(T)$, and Γ_i maps to $c^{-B(i)}\mathbf{F}^e(T)$. We will extend $\psi_e^{(i)}$ to a map from $S^{(i+1)}$ such that each $U_j^{(i)}$ maps to $c^{-\beta(i+1)}\mathbf{F}^e(T)$, and Γ_{i+1} maps to $c^{-B(i+1)}\mathbf{F}^e(T)$.

In order to simplify notation, we drop many of the (i) labels on parameters. Then

$$S^{(i+1)} = \frac{S^{(i)}[U_1, \dots, U_k]}{s - \sum_{j=1}^{k} x_j U_j}.$$

Since s and the s_j (in the relation $x_{k+1}s = \sum_{j=1}^{k} x_j s_j$ in $S^{(i)}$) are in Γ_i, we can write

$$\psi_e^{(i)}(s) = c^{-B(i)}\sigma \qquad \text{and} \qquad \psi_e^{(i)}(s_j) = c^{-B(i)}\sigma_j, \tag{6.2.10}$$

where σ and the σ_j are elements in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Hence,

$$x_{k+1}^q \psi_e^{(i)}(s) = \sum_{j=1}^{k} x_j^q \psi_e^{(i)}(s_j)$$

in $\mathbf{F}^e(T)_c$. Multiplying through by $c^{B(i)}$ yields

$$x_{k+1}^q \sigma = \sum_{j=1}^{k} x_j^q \sigma_j$$

in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Using our (weak) colon-killer c, we have

$$c\sigma = \sum_{j=1}^{k} x_j^q \tau_j,$$

where τ_j is an element in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Therefore,

$$\psi_e^{(i)}(s) = \sum_{j=1}^{k} x_j^q (c^{-B(i)-1}\tau_j)$$

in $\mathbf{F}^e(T)_c$.

We now have a well-defined map $\psi_e^{(i+1)} : S^{(i+1)} \to \mathbf{F}^e(T)_c$ extending $\psi_e^{(i)}$ given by

$$\psi_e^{(i+1)}(U_j) = c^{-B(i)-1}\tau_t = c^{-\beta(i+1)}\tau_t$$

such that the U_j map to $c^{-\beta(i+1)}\mathbf{F}^e(T)$, and Γ_{i+1} maps to $c^{-B(i+1)}\mathbf{F}^e(T)$ since

$$B(i+1) = \beta(i+1)b(i+1) + B(i)$$

and these elements can be written as polynomials in the U_j of degree at most $b(i+1)$ with coefficients in Γ_i.

We can finally conclude that, for all $q = p^e$, there exists a map

$$\psi_e^{(t)} : S^{(t)} \to \mathbf{F}^e(T)_c$$

such that the equation (6.2.9) that puts $1 \in mS^{(t)}$ maps to

$$1 = \sum_{j=1}^{n} r_j^q \psi_e^{(t)}(w_j).$$

If we let $B := B(t)$, then each $\psi_e^{(t)}(w_j)$ is in $c^{-B}\mathbf{F}^e(T)$ as Γ_t contains these elements. Multiplying through by c^B, we see that $c^B \in m^{[q]}T$, for all $q \geq 1$, where B is independent of q, which implies that $c^B \in m^k T$, for all $k \geq 1$. Since c is a (weak) durable colon-killer, we have a contradiction. Therefore, no such finite sequence of modifications of S is bad.

In the case of the durable colon-killer, we see that S is a seed over R. In the case of a weak colon-killer, S maps to an S-algebra S', where a single system of parameters of R is a regular sequence on S'. By the result [Bar-Str, Theorem 1.7] of Bartijn and Strooker, the separated completion of S' with respect to the maximal ideal of R is a big Cohen-Macaulay R-algebra. Therefore, S is a seed in this case as well. □

Later we will use this result as a piece of the proof that integral extensions of seeds are seeds. We will also use durable colon-killers to obtain our results in the sixth section concerning when the seed property is preserved by base change.

6.3 Minimal Seeds

We will now introduce the class of *minimal seeds*. This class will help us gain more insight into the class of big Cohen-Macaulay algebras to which a particular seed can be mapped.

Definition 6.3.1. For a Noetherian local ring (R, m), an R-algebra S is a *minimal seed* if S is a seed over R and no proper homomorphic image of S is a seed over R.

Example 6.3.2. (1) If R is a Cohen-Macaulay ring, then R is a minimal seed.

(2) If R is an excellent local domain, then R^+ is a big Cohen-Macaulay algebra over R. The ring R^+ is also a minimal seed. Indeed, let $J \subseteq R^+$, with $s \neq 0$ in J. Therefore, s satisfies a minimal monic polynomial $x^k + r_1 x^{k-1} + \cdots + r_{k-1}x + r_k$,

with coefficients $r_i \in R$. Since the polynomial is minimal and R^+ is a domain, we have a nonzero element $r_k \in J \cap R$. Therefore, r_k kills R^+/J, and so R^+/J cannot be a seed, which shows that R^+ is a minimal seed as claimed.

We also point out the following easy, but useful, fact about minimal seeds.

Lemma 6.3.3. *A seed S over a local Noetherian ring R is a minimal seed if and only if every map from S to a big Cohen-Macaulay R-algebra B is injective if and only if every map to a seed over R is injective.*

Proof. If S is minimal, then since no homomorphic image of S can be a seed, S must be isomorphic to its image in every big Cohen-Macaulay algebra to which it maps. Conversely, if S is not minimal, then there is a nonzero ideal $I \subseteq S$ such that $S/I \to B$, for some big Cohen-Macaulay algebra B. Thus, $S \to B$ is not injective.

The second claim is equally easy to see. $\square$

A very important question one should ask about minimal seeds is whether or not every seed maps to a minimal seed. Whereas it is currently unknown whether solid algebras that are not Noetherian map to minimal solid algebras, we are able to show that every seed maps to a minimal seed.

Proposition 6.3.4. *Let R be a local Noetherian ring, and let S be a seed over R. Then S/I is a minimal seed for some ideal $I \subseteq S$.*

Proof. Let Σ be the set of all ideals J of S such that S/J is a seed. If Σ contains a maximal element I, then S/I will be a minimal seed. Let $J_1 \subset J_2 \subset \cdots$ be a chain of ideals in Σ, and let $J = \bigcup_k J_k$. Then $S/J = \varinjlim_k S/J_k$, and since each S/J_k is a seed, Lemma 6.1.2 implies that S/J is a seed as well. By Zorn's Lemma, Σ has a maximal element I. $\square$

Now that we know minimal seeds exist, we would like to know whether they are domains or not, as minimal Noetherian solid algebras are domains. After dealing with integral extensions of seeds, we will prove in Section 6.5 that in positive characteristic minimal seeds are domains. In the meantime, we will point out that minimal seeds are reduced in positive characteristic.

Proposition 6.3.5. *Let S be a minimal seed over a local ring R of positive characteristic. Then S is a reduced ring.*

Proof. By Lemma 6.1.3, there exists a reduced big Cohen-Macaulay algebra B such that $S \to B$. Since S is minimal, Lemma 6.3.3 implies that $S \hookrightarrow B$. As B has no nilpotents, neither does S. □

The fact that all seeds map to a reduced seed in positive characteristic will be helpful in the next section, when we demonstrate that integral extensions of seeds are still seeds. In turn, that result will eventually help us prove that each seed maps to a domain seed and also to a big Cohen-Macaulay algebra that is a domain.

6.4 Integral Extensions of Seeds

The primary goal of this section is to prove that integral extensions of seeds are seeds in positive characteristic. Since all integral extensions are direct limits of module-finite extensions, with Lemma 6.1.2 we can concentrate on module-finite extensions of seeds. We begin the argument by proving that the problem can be reduced to a much more specific problem, which we attack by constructing a durable colon-killer in a certain module-finite extension of a big Cohen-Macaulay algebra. Our first reduction of the problem will be that we can assume we are considering a module-finite extension of a big Cohen-Macaulay algebra that is reduced, quasilocal, and m-adically separated.

Lemma 6.4.1. *Let (R, m) be a local Noetherian ring of positive characteristic, and let S be a seed with T a module-finite extension of S. Suppose that any module-finite extension of a reduced, quasilocal, m-adically separated big Cohen-Macaulay algebra is a seed. Then T is a seed.*

Proof. By Propositions 6.3.4 and 6.3.5, S/I is a minimal reduced seed for some ideal I. Since S/I is reduced, I is a radical ideal and so is an integrally closed ideal. Therefore, $IT \cap S = I$ so that S/I injects into T/IT, which is thus a module-finite extension of S/I. Since T is a seed if T/IT is a seed, we can now assume that S is a reduced seed.

By Lemma 6.3.3 and Proposition 6.1.7, there exists a commutative square

$$\begin{tikzcd} T \arrow[r] & C \\ S \arrow[u, hook] \arrow[r, hook] & B \arrow[u] \end{tikzcd},$$

where B is a reduced, quasilocal, and m-adically separated big Cohen-Macaulay algebra, and $C := T \otimes_S B$. If we can show that the vertical map $B \to C$ is injective, then C will be a module-finite extension of B, and our hypotheses will imply that C and T are seeds. Since B is reduced, the next lemma allows us to reach our goal. $\square$

Lemma 6.4.2. *If S is a ring, T is an integral extension of S, and B is a reduced extension of S, then B injects into $C := T \otimes_S B$.*

Proof. We will first prove the claim in the case that S, T, and B are all domains. Let K be the algebraic closure of the fraction field of S, and let L be the algebraic closure of the fraction field of B. Since T is an integral extension domain of S, we have the following diagram:

$$\begin{tikzcd} K \arrow[r, hook] & L \\ T \arrow[u, hook] & \\ S \arrow[u, hook] \arrow[r, hook] & B. \arrow[uu, hook] \end{tikzcd}$$

Under the injection $K \hookrightarrow L$, the ring T maps isomorphically to some subring T' of L. Now let C' be the S-subalgebra of L generated by B and T'. Since $C = T \otimes_S B$, we have a (surjective) map $C \to C'$ and a diagram

$$\begin{tikzcd} & & C' \\ T \arrow[r] \arrow[urr, hook, bend left] & C \arrow[ur] & \\ S \arrow[u, hook] \arrow[r, hook] & B \arrow[u] \arrow[uur, hook, bend right] & \end{tikzcd}$$

Since B injects into C', B also injects into C.

For the general case, it will suffice to show that the kernel of $B \to C$ is contained in every prime ideal of B since B is reduced. Let $\mathfrak{p}$ be a prime of B, and let $\mathfrak{p}_0 := \mathfrak{p} \cap S$. Since T is integral over S, there exists a prime $\mathfrak{q}_0$ of T lying over $\mathfrak{p}_0$. If we put $D := T/\mathfrak{q}_0 \otimes_{S/\mathfrak{p}_0} B/\mathfrak{p}$, then we obtain the following commutative diagram:

$$\begin{tikzcd} & T/\mathfrak{q}_0 \arrow[rr] & & D \\ T \arrow[ur] \arrow[rr] & & C \arrow[ur] & \\ & S/\mathfrak{p}_0 \arrow[uu, hook] \arrow[rr, hook] & & B/\mathfrak{p} \arrow[uu] \\ S \arrow[uu, hook] \arrow[ur] \arrow[rr, hook] & & B \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Since $T/\mathfrak{q}_0$ is still an integral extension of $S/\mathfrak{p}_0$ and $B/\mathfrak{p}$ is a domain extension of $S/\mathfrak{p}_0$, the domain case of the proof shows that $B/\mathfrak{p}$ injects into D. Therefore, if b is in the kernel of $B \to C$, then b is in the kernel of $B \to D$, which implies that $b \in \mathfrak{p}$, as desired. $\square$

Now, in order to show that module-finite extensions of seeds are still seeds, it suffices to consider a module-finite extension C of a reduced, quasilocal, m-adically separated big Cohen-Macaulay algebra B. To finish our argument that module-finite extensions of seeds are seeds, will show that we can extend the map $B \hookrightarrow C$ to another module-finite extension $B^{\#} \hookrightarrow C^{\#}$ such that $B^{\#}$ is still a reduced, quasilocal, m-adically separated big Cohen-Macaulay algebra. In addition, our new rings will have the property that there exists a nonzero element $b \in B^{\#}$ such that b multiplies $C^{\#}$ into a finitely generated free $B^{\#}$-submodule of $C^{\#}$. We will then finally show that b is a durable colon-killer in $C^{\#}$ so that Theorem 6.2.8 implies that C is a seed.

We start the process by showing we can adjoin indeterminates and then localize our ring B without losing any of its key properties.

Lemma 6.4.3. *If $(B, \mathfrak{p})$ is a reduced, quasilocal, m-adically separated big Cohen-Macaulay R-algebra, where (R, m) is a local Noetherian ring, then the ring*

$$B^{\#(n,s)} := B[t_{ij} \,|\, i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]},$$

defined for some $n, s \in \mathbb{N}$, is also a reduced, quasilocal, m-adically separated big Cohen-Macaulay R-algebra.

Proof. For the duration of the proof, n and s will be fixed, so we will simply write $B^{\#}$ instead of $B^{\#(n,s)}$. We will let $\mathbf{t}$ denote the set of all t_{ij}.

Since B is reduced, $B[\mathbf{t}]$ is reduced after adjoining indeterminates. By Lemma 6.1.5, $B^{\#}$ will also be reduced. As $\mathfrak{p}$ is prime in B, the extension of $\mathfrak{p}$ to $B[\mathbf{t}]$ is also prime so that it makes sense to localize at this ideal and end up with $B^{\#}$ quasilocal.

The construction of $B^{\#}$ implies that $B^{\#}$ is a faithfully flat extension of B. Therefore, for any ideals I and J of B, we have $IB^{\#} :_{B^{\#}} JB^{\#} = (I :_B J)B^{\#}$ (see [N2, Theorem 18.1]). This fact implies that every system of parameters of R will be a possibly improper regular sequence on $B^{\#}$, because B is a big Cohen-Macaulay algebra. Moreover, the faithful flatness also implies that $mB^{\#} \neq B^{\#}$ as $mB \neq B$. Hence, $B^{\#}$ is a big Cohen-Macaulay R-algebra.

To show that $B^{\#}$ is also m-adically separated will take a little bit more effort. Suppose that an element $F \in B^{\#}$ is in $m^N B^{\#}$, for every N. Multiplying through by its denominator, we obtain such an element from $B[\mathbf{t}]$, so that we may assume without loss of generality that F is a polynomial in $B[\mathbf{t}]$. Thus, for every N, there exists $G_N \notin \mathfrak{p}B[\mathbf{t}]$ such that $G_N F \in m^N B[\mathbf{t}]$. It suffices to show that any polynomial $G \notin \mathfrak{p}B[\mathbf{t}]$ is not a zerodivisor modulo $m^N B[\mathbf{t}]$, for any N. If we put $D := B/m^N B$, then the image $\overline{G}$ of G in $D[\mathbf{t}]$ is a polynomial not in $\mathfrak{p}D[\mathbf{t}]$, i.e., $\overline{G}$ is a polynomial whose coefficients generate the unit ideal of D. To finish, it suffices to apply the following general lemma. $\square$

Lemma 6.4.4. *If D is any ring and G is a polynomial in $D[t_\lambda \mid \lambda \in \Lambda]$ such that the coefficients of G generate the unit ideal in D, then G is not a zerodivisor.*

Proof. As in the previous proof, we will let $\mathbf{t}$ denote the set of all t_λ. Suppose that $GH = 0$, for some $H \in D[\mathbf{t}]$. If A is the prime ring of D, then let D' be the A-subalgebra of D generated by the coefficients of G and H and by the elements in a relation showing that 1 is in the ideal generated by the coefficients of G. Therefore, $GH = 0$ in $D'[\mathbf{t}]$, and the coefficients of G still generate the unit ideal in D', so without loss of generality we may assume that D is Noetherian.

The Noetherian case follows from Corollary 2 on p. 152 of [Mat]. $\square$

Lemma 6.4.5. *If $(B, \mathfrak{p})$ is a quasilocal ring, and $B^{\#(n,s)} = B[t_{ij} \,|\, i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]}$, for some n and some s, then for any $k \leq n$,*

$$B^{\#(n,s)} \cong (B^{\#(k,s)})^{\#(n-k,s)}.$$

Proof. Let $\mathbf{x}$ denote the indeterminates t_{ij} such that $1 \leq i \leq k$ and $1 \leq j \leq s$, and let $\mathbf{y}$ denote the remaining indeterminates t_{ij}. Let $C := B^{\#(k,s)} = B[\mathbf{x}]_{\mathfrak{p}B[\mathbf{x}]}$, Q be the maximal ideal of C, and $D := (B^{\#(k,s)})^{\#(n-k,s)} = C[\mathbf{y}]_{QC[\mathbf{y}]}$.

Since D is a B-algebra, there exists a unique ring homomorphism $B[\mathbf{x}, \mathbf{y}] \to D$ that maps the indeterminates t_{ij} to their natural images in D. We claim that the units in $B[\mathbf{x}, \mathbf{y}]$ map to units in D under this map. Indeed, if $f(\mathbf{x}, \mathbf{y})$ is a unit in $B[\mathbf{x}, \mathbf{y}]$, then f has some coefficient that is in $B \setminus \mathfrak{p}$. If we rewrite

$$f(\mathbf{x}, \mathbf{y}) = g_k(\mathbf{x})\mathbf{y}^k + \cdots + g_1(\mathbf{x})\mathbf{y} + g_0(\mathbf{x}),$$

then some $g_i(\mathbf{x})$ is in $C \setminus Q$ so that the image of f is in not in the maximal ideal of D. We therefore have a ring homomorphism $\phi : B^{\#(n,s)} \to D$.

We further claim that ϕ is an isomorphism. Using the previous lemma, it is easy to verify that ϕ is injective. It is also routine to check that ϕ is surjective. $\square$

Using the construction $B^{\#(n,s)} = B[t_{ij} \,|\, i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]}$, where $(B, \mathfrak{p})$ is a quasilocal ring, we also define

$$^{\#(n,s)}M := B^{\#(n,s)} \otimes_B M,$$

for any B-module M. Since $B^{\#(n,s)}$ is faithfully flat over B, when $M = C$ is a module-finite extension of B, we also have that $^{\#(n,s)}C$ is a module-finite extension of $B^{\#(n,s)}$. The utility of this operation is that the adjunction of indeterminates will allow us to find $B^{\#(n,s)}$-linear combinations of the module generators of $^{\#(n,s)}C$ such that these elements are in "general position" and, as a result, span a $B^{\#(n,s)}$-free submodule of ${}^{\#(n,s)}C$. Moreover, the quotient of ${}^{\#(n,s)}C$ by this free submodule will be killed by a nonzero element b of $B^{\#(n,s)}$. After we show the existence of such an element for a particular value of n, we will show that the element b is a durable colon-killer in ${}^{\#(n,s)}C$ when B is a reduced, quasilocal, m-adically separated big Cohen-Macaulay R-algebra over a local Noetherian ring (R, m).

Lemma 6.4.6. *Let B be a reduced quasilocal ring, and let M be a finitely generated B-module generated by $m_1, \ldots, m_s$ in M. Then there exists $k \leq s$ such that $b({}^{\#(k,s)}M) \subseteq G$, where b is a nonzero element of $B^{\#(k,s)}$ and G is a finitely generated free $B^{\#(k,s)}$-submodule of ${}^{\#(k,s)}M$.*

Proof. Throughout the proof, define

$$g_i := t_{i1}m_1 + \cdots + t_{is}m_s$$

in any $B^{\#(n,s)}$, where $i \leq n$. Note that there exists a maximum $0 \leq n \leq s$ such that the set $\{g_1, \ldots, g_n\}$ generates a $B^{\#(n,s)}$-free submodule of ${}^{\#(n,s)}M$, where $B^{\#(0,s)} = B$, since ${}^{\#(n,s)}M$ has s generators. If $\alpha = (t_{ij})_{1\leq i,j\leq s}$, then $\det(\alpha)$ is not in the unique maximal ideal of $B^{\#(s,s)}$, so that α is an invertible matrix. As $m_1, \ldots, m_s$ generate ${}^{\#(s,s)}M$ over $B^{\#(s,s)}$ and α is invertible, $g_1, \ldots, g_s$ also generate ${}^{\#(s,s)}M$. If the g_i are linearly independent over $B^{\#(s,s)}$, then ${}^{\#(s,s)}M$ is a free module, and we can use $k = s$, $b = 1$, and $G = {}^{\#(s,s)}M$ to fulfill our claim.

Otherwise, the maximum value n is strictly less than s, and we put $k := n+1$. In this case, there exists a nonzero $b' \in B^{\#(k,s)}$ such that $b'g_k \in (g_1, \ldots, g_{k-1})({}^{\#(k,s)}M)$. Indeed, since n was chosen to be a maximum and $k = n + 1$, there must be a nontrivial relation $b'g_k = b_1g_1 + \cdots b_{k-1}g_{k-1}$ in ${}^{\#(k,s)}M$. If $b' = 0$, then we have a nontrivial relation on $g_1, \ldots, g_{k-1}$. As $B^{\#(k,s)} \cong (B^{\#(k-1,s)})^{\#(1,s)}$ (by Lemma 6.4.5), we see that $B^{\#(k,s)}$ is faithfully flat over $B^{\#(k-1,s)}$, and so the nontrivial relation on $g_1, \dots, g_{k-1}$ in ${}^{\#(k,s)}M$ implies that there is a nontrivial relation on $g_1, \dots, g_{k-1}$ in ${}^{\#(k-1,s)}M$, a contradiction. Hence, b' is nonzero as claimed. Notice that the same argument implies that $g_1, \dots, g_{k-1}$ still generate a finitely generated free submodule G of ${}^{\#(k,s)}M$.

We now claim that there exists a nonzero $b \in B^{\#(k,s)}$ such that $b({}^{\#(k,s)}M) \subseteq G$. If we put $M_0 := ({}^{\#(k,s)}M)/G$, and replace $m_1, \dots, m_s$ and g_k by their images in M_0, then b' kills g_k. We intend to show that the annihilator of M_0 cannot be zero. Suppose to the contrary that no nonzero element of $B^{\#(k,s)}$ kills M_0. Then we have an injective map $B^{\#(k,s)} \hookrightarrow M_0^{\oplus s}$ defined by $b \mapsto (bm_1, \dots, bm_s)$.

After clearing the denominator on b', we may assume that b' is a polynomial in $B[\mathbf{t}]$, where $\mathbf{t}$ denotes the set of all t_{ij}, with $i \le k$ and $j \le s$. Write $b' = \sum_\nu c_\nu \mathbf{t}^\nu$, where ν is an $n \times s$ matrix of integers, and each c_ν is in B. Let A_0 be the prime ring of B, and let A be the A_0-subalgebra of B (finitely) generated by the nonzero c_ν. Then A is Noetherian and reduced. If we let $\mathfrak{q}$ be the contraction of $\mathfrak{p}$ to A and replace A by the local ring $A_\mathfrak{q}$, then $(A, \mathfrak{q})$ is a reduced local Noetherian subring of B. We then obtain an injective map $A^\# \hookrightarrow B^{\#(k,s)}$, where $A^\#$ denotes $A^{\#(k,s)}$.

Since b' is in $A^\#$ (and still nonzero), we can define an $A^\#$-module by

$$N := \frac{A^\# m_1 \oplus \cdots \oplus A^\# m_s}{b'(t_{k1}m_1 + \cdots + t_{ks}m_s)}.$$

There is then a natural map $N \to M_0$ that induces a commutative square:

$$\begin{tikzcd} B^{\#(k,s)} \arrow[r, hook] & M_0^{\oplus s} \\ A^\# \arrow[u, hook] \arrow[r] & N^{\oplus s}. \arrow[u] \end{tikzcd}$$

This implies that the map $A^\# \to N^{\oplus s}$ defined by $a \mapsto (am_1, \dots, am_s)$ is injective (which also shows that $N \neq 0$). Therefore, we may now assume without loss of generality that B is a reduced, local Noetherian ring.

As B is reduced and Noetherian, B has finitely many minimal primes $Q_1, \ldots, Q_h$ such that $\bigcap_i Q_i = 0$. Since b' is a nonzero polynomial in $B^{\#(k,s)}$, some coefficient of b' is not in some minimal prime Q. Thus, the image of b' is still nonzero in $(B_Q)^{\#(k,s)}$. Moreover, if $M_0' := (B_Q)^{\#(k,s)} \otimes_{B^{\#(k,s)}} M_0$, then M_0' is a finitely generated $(B_Q)^{\#(k,s)}$-module with $b'(t_{k1}m_1 + \cdots + t_{ks}m_s) = 0$ and with an injection $(B_Q)^{\#(k,s)} \hookrightarrow (M_0')^{\oplus s}$, since $(B_Q)^{\#(k,s)} \cong (B^{\#(k,s)})_{QB^{\#(k,s)}}$. (Again, this fact implies that M_0' is nonzero.) Since B is reduced and Q is minimal, B_Q is a field, and so we can now assume that $B = K$ is a field.

In this final case, $K^{\#(k,s)} \cong K(\mathbf{t})$, and M_0 is a nonzero module over a field, so that M_0 is a nonzero free $K^{\#(k,s)}$-module. Therefore, if $b'(t_{k1}m_1 + \cdots + t_{ks}m_s) = 0$ in M_0, then $t_{k1}m_1 + \cdots + t_{ks}m_s = 0$ in M_0. This is impossible, however, since the t_{ij} are algebraically independent.

The resulting contradiction implies that M_0 is killed by some nonzero element $b \in B^{\#(k,s)}$ in our original set-up, and since M_0 was originally $({}^{\#(k,s)}M)/G$, where G is free over $B^{\#(k,s)}$, we are finished. $\square$

We are now ready to show that module-finite extensions of sufficiently nice big Cohen-Macaulay algebras are indeed seeds. As mentioned above, the key fact will be that the element b constructed in the previous lemma is a durable colon-killer in the modification ${}^{\#(k,s)}C$.

Lemma 6.4.7. *Let (R, m) be a local Noetherian ring, and let B be a reduced, quasilocal, m-adically separated big Cohen-Macaulay R-algebra. If C is a module-finite extension of B, then C is a seed.*

Proof. By Lemma 6.4.3 and the remarks made before the previous lemma, for any

k, we have a commutative square:

$$\begin{tikzcd} B^{\#(k,s)} \arrow[r, hook] & {}^{\#(k,s)}C \\ B \arrow[u] \arrow[r, hook] & C, \arrow[u] \end{tikzcd}$$

where the top map is also a module-finite extension of a reduced, quasilocal, m-adically separated big Cohen-Macaulay R-algebra, and C is generated by s elements as a B-module. After applying the previous lemma, we may assume that there exists a nonzero element $b \in B$ such that b multiplies C into a finitely generated free B-submodule G. In order to see that C is a seed, we show that b is a durable colon-killer in C and then apply Theorem 6.2.8.

Indeed, let $x_1, \ldots, x_{t+1}$ be part of a system of parameters in R and suppose that $u \in (x_1, \ldots, x_t)C :_C x_{t+1}$. Then by construction, $bux_{t+1} \in (x_1, \ldots, x_t)G$, so as an element of G, we have $(bu) \in (x_1, \ldots, x_t)G :_G x_{t+1}$. Since B is a big Cohen-Macaulay R-algebra and since G is a free B-module, G is clearly a big Cohen-Macaulay R-module. Hence, $bu \in (x_1, \ldots, x_t)G \subseteq (x_1, \ldots, x_t)C$ as G is a submodule of C.

Now, if $b^N \in \bigcap_t m^t C$, for some N, then $b^{N+1} \in \bigcap_t m^t G$. Since B is m-adically separated, $\bigcap_t m^t B = 0$, and since G is free over B, we also have $\bigcap_t m^t G = 0$. As G is a submodule of C and the map $B \to C$ is an injection, $b^{N+1} = 0$ in B, but B reduced implies that $b = 0$, a contradiction. Therefore, b is a durable colon-killer, and C is a seed by Theorem 6.2.8. $\square$

We have now gathered together all of the tools that we will need to prove the primary result of this section.

Theorem 6.4.8. *Let (R, m) be a local Noetherian ring of positive characteristic. If S is a seed and T is an integral extension of S, then T is a seed.*

Proof. By Lemma 6.1.2, we may assume that T is a module-finite extension of S because integral extensions are direct limits of module-finite extensions. By Lemma 6.4.1, we may assume that $S = B$ is a reduced, quasilocal, m-adically separated big Cohen-Macaulay algebra. Finally, Lemma 6.4.7 implies that T is a seed. □

Remark 6.4.9. We feel it is worthwhile to point out that the hypothesis that our base ring has positive characteristic is only required in two places: (1) the existence of a durable colon-killer implies that an algebra is a seed, and (2) all seeds map to a reduced big Cohen-Macaulay algebra. These facts have proofs that rely heavily on the use of the Frobenius endomorphism, but are the only two that we can prove only in positive characteristic.

We can also view the above theorem as a generalization of the existence of big Cohen-Macaulay algebras over complete local domains of positive characteristic.

Corollary 6.4.10 (Hochster-Huneke)**.** *If R is a complete local domain of positive characteristic, then there exists a big Cohen-Macaulay algebra B over R.*

Proof. By the Cohen structure theorem, R is a module-finite extension of a regular local ring A. Since A is clearly a seed over itself, Theorem 6.4.8 implies that R is a seed over A as well. Let B be a big Cohen-Macaulay algebra over A such that B is also an R-algebra. By Corollary 6.2.6, B is also big Cohen-Macaulay over R. □

6.5 More Properties of Seeds

In this section, we will show that all seeds in positive characteristic can be mapped to quasilocal big Cohen-Macaulay algebra domains that are absolutely integrally closed and m-adically separated. We start off the section by delivering the promised proof that minimal seeds are domains. First, we show that we can reduce to the case of a finitely generated minimal seed.

Lemma 6.5.1. *Let R be a local Noetherian ring. If all finite type minimal seeds are domains, then all minimal seeds are domains.*

Proof. Let S be an arbitrary minimal seed over R. Then $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$, where Λ indexes the set of all finitely generated subalgebras of S. Suppose that S is not a domain and that $ab = 0$ in S with $a, b \neq 0$. Since S is a minimal seed, S/aS and S/bS are not seeds. Let $\Lambda(a)$ and $\Lambda(b)$ be the subsets of Λ indexing all finitely generated subalgebras of S that contain a and b, respectively. Then $S/aS = \varinjlim_{\lambda\in\Lambda(a)} S_\lambda/aS_\lambda$, with a similar result for S/bS. Since S/aS and S/bS are not seeds, Lemma 6.1.2 implies that there exists an S_α containing a and an S_β containing b such that S_α/aS_α and S_β/bS_β are not seeds. Therefore, there exists a common S_γ containing a and b such that S_γ is not a seed modulo aS_γ nor modulo bS_γ. We can enlarge S_γ further and so also assume without loss of generality that $ab = 0$ in S_γ, since $ab = 0$ in S. Since S is a seed, S_γ is a seed. Since S_γ is also finitely generated as an R-algebra, S_γ maps onto a finitely generated minimal seed T. Therefore, $ab = 0$ in T, and as T is a domain by hypothesis, $a = 0$ or $b = 0$ in T. Suppose without loss of generality that $a = 0$. This implies that the map $S_\gamma \to T$ factors through S_γ/aS_γ, which is not a seed and so cannot map to any seed. We have a contradiction, and so S is a domain after all. □

Proposition 6.5.2. *Let R be a local Noetherian ring of positive characteristic p. If S is a minimal seed over R, then S is a domain.*

Proof. By the previous lemma, we can assume that S is finitely generated over R. By Proposition 6.3.5, we have that S is Noetherian and reduced. Let $\overline{S}$ be the normalization of S in its total quotient ring. Then $\overline{S}$ is a finite direct product of normal domains by Serre's Criterion (see [E, Theorem 11.5]) and is an integral extension of S. By our main result of the last section, Theorem 6.4.8, $\overline{S}$ is also a seed. Since $\overline{S}$ is a seed and a finite product $D_1 \times \cdots \times D_t$ of domains, we will be done once we have proven the following lemma. $\square$

Lemma 6.5.3. *Let (R, m) be a Noetherian local ring. If $S = S_1 \times \cdots \times S_t$, then S is a seed over R if and only if S_i is a seed over R, for some i.*

Proof. Clearly, if some S_i is a seed, then S is also a seed. Suppose then that S is a seed, but no S_i is a seed. Since S is a direct product, if $S \to B$, a big Cohen-Macaulay algebra, then $B \cong B_1 \times \cdots \times B_t$, where each B_i is an S_i-algebra. We first claim that each B_i is a possibly improper big Cohen-Macaulay algebra. Indeed, let $x_{k+1}b = \sum_{j=1}^{k} x_j b_j$ be a relation in B_i on a partial system of parameters $x_1, \ldots, x_{k+1}$ from R, and let e_i be the idempotent associated to B_i in B. Therefore, $x_{k+1}(be_i) = \sum_{j=1}^{k} x_j(b_j e_i)$ is a relation in B, and so $be_i = \sum_{j=1}^{k} x_j c_j$, for elements c_j in B. Multiplying this equation by e_i yields $be_i = \sum_{j=1}^{k} x_j(c_j e_i)$ since $e_i^2 = e_i$. If we let c_j' be the image of $c_j e_i$ in B_i, for all j, then $b = \sum_{j=1}^{k} x_j c_j'$ in B_i, as claimed.

Now, if, as assumed, each S_i is not a seed, then $1 \in mB_i$, for all i. Thus $e_i \in mB$, for all i, and so $1 = \sum_i e_i \in mB$, a contradiction. Therefore, some S_i must be a seed if S is a seed. $\square$

As a corollary, we will show that each seed maps to a big Cohen-Macaulay algebra that is also a domain. In order to accomplish this goal, we must first demonstrate that a domain seed can be modified into a big Cohen-Macaulay algebra domain. In [HH7, Section 3], Hochster and Huneke prove that if S is a seed, then S maps to a big Cohen-Macaulay algebra B constructed as a very large direct limit of sequences of algebra modifications. To obtain our result, we will use a different direct limit system of algebra modifications to construct a big Cohen-Macaulay algebra B.

We will use the "$\mathfrak{A}$-transform"

$$\Theta = \Theta(x, y; S) := \{u \in S_{xy} \,|\, (xy)^N u \subseteq \mathrm{Im}(S \to S_{xy}), \text{ for some } N\}.$$

See [N1, Chapter V] and [Ho3, Section 12] for an introduction to the properties of Θ. The most useful property for us is that if x, y form part of a system of parameters in a local Noetherian ring R and S is a seed over R, then x, y become a regular sequence on Θ (see [Ho3, Lemma 12.4]). As a result any map from S to a big Cohen-Macaulay R-algebra B factors through Θ. Indeed, let ϕ be the map $S \to \Theta$ and let $f : S \to B$. Given $u \in \Theta$, $x^N u = \phi(s_1)$ and $y^N u = \phi(s_2)$, where $s_1, s_2 \in S$. Since $y^N s_1 - x^N s_2$ is in the kernel of ϕ, there exists $M \geq 0$ such that $x^M y^{M+N} s_1 = x^{M+N} y^M s_2$ in S. If $f(s_1) = b_1$ and $f(s_2) = b_2$, then $x^M y^{M+N} b_1 = x^{M+N} y^M b_2$ in B, and so $y^N b_1 = x^N b_2$ as $(xy)^M$ is not a zerodivisor in B. Moreover, since B is a big Cohen-Macaulay algebra, $b_1 = x^N b_1'$ and $b_2 = y^N b_2'$, where $b_1' = b_2'$ as $(xy)^N b_1' = (xy)^N b_2'$ in B. We can then extend the map $f : S \to B$ to $\Theta \to B$ by mapping u to $b_1' = b_2'$. It is then straightforward to check that this map is a well-defined ring homomorphism.

Suppose now that S is a seed over a local Noetherian ring R, and let

$$T = \frac{S[U_1, \ldots, U_k]}{(s - s_1 U_1 - \cdots - s_k U_k)}$$

be an algebra modification of S, where $x_{k+1} s = x_1 s_1 + \cdots + x_k s_k$ is a relation in S on a partial system of parameters $x_1, \ldots, x_{k+1}$ in R. When $k \geq 2$, we also have an induced relation on $x_1, \cdots, x_{k+1}$ in $\Theta = \Theta(x_1, x_2; S)$ so that

$$T' = \frac{\Theta[U_1, \ldots, U_k]}{(s - s_1 U_1 - \cdots - s_k U_k)}$$

is an algebra modification of Θ over R. We will call T' an *enhanced algebra modification of S over R induced by the relation $x_{k+1} s = x_1 s_1 + \cdots + x_k s_k$.*

With our remarks above about how any map from S to a big Cohen-Macaulay algebra B factors through Θ, we obtain a commutative diagram

$$\begin{tikzcd} \Theta \arrow[r] & T' \arrow[dr] & \\ S \arrow[u] \arrow[r] & T \arrow[u] \arrow[r] & B \end{tikzcd}$$

which shows that maps from algebra modifications of seeds to big Cohen-Macaulay algebras factor through the induced enhanced modification of S when T is a modification with respect to a relation on 3 or more parameters from R. With the use of this factorization, we can adapt the process described in [HH7, Section 3] to construct a big Cohen-Macaulay algebra from a given seed as a very large direct limit of enhanced modifications and ordinary modifications with respect to relations on 1 or 2 parameters.

The following lemma shows that enhanced algebra modifications preserve the properties of being reduced or a domain. This fact will show us that domain seeds and reduced seeds (in any characteristic) can be mapped to big Cohen-Macaulay algebras that are domains or reduced, respectively.

Lemma 6.5.4. *Let (R, m) be a local Noetherian ring, and let S be a domain (resp., reduced). If T is an enhanced algebra modification of S over R, then T is also a domain (resp., reduced).*

Proof. Let T be induced by the relation $x_{k+1}s = x_1s_1 + \cdots + x_ks_k$ in S, where $k \geq 2$. Then x_1, x_2 forms a possibly improper regular sequence on $\Theta = \Theta(x_1, x_2; S)$ (see [Ho3, Lemma 12.4]), and so $x_1, s - x_1U_1 - \cdots - x_kU_k$ is a possibly improper regular sequence on $\Theta[U_1, \ldots, U_k]$. (Any polynomial $f(U)$ that kills $s - x_1U_1 - \cdots - x_kU_k$ modulo x_1 has a highest degree term as a polynomial in U_2, but this term is killed by x_2 modulo x_1. Since x_2 is not a zerodivisor modulo x_1, the term must be divisible by

x_1. Hence, $f(U)$ is divisible by x_1, and $s-x_1U_1-\cdots-x_kU_k$ is not a zerodivisor modulo x_1.) Therefore, x_1 is not a zerodivisor on $\Theta[U_1,\ldots,U_k]/(s-x_1U_1-\cdots-x_kU_k)$. The result now follows from the following short lemma. □

Lemma 6.5.5. *Let A be a domain (resp., reduced). If a and x are elements of A such that x is not a zerodivisor on $A' := A[U]/(a-xU)$, then A' is also a domain (resp., reduced).*

Proof. Since x is not a zerodivisor on A', we have an inclusion $A' \hookrightarrow (A')_x$ so that it suffices to show $(A')_x$ is a domain (resp., reduced). It is, however, easy to verify that $(A')_x \cong A_x$ via the map that sends U to a/x (even without any hypotheses on A or x). Since A is a domain (resp., reduced), so is A_x. □

As promised, we now prove that one can modify a domain or reduced seed into a big Cohen-Macaulay algebra with the same property. Note that the result is characteristic free and thus independent of the reduced result proved earlier in the first section of the chapter.

Proposition 6.5.6. *Let R be a local Noetherian ring. If S is a seed and a domain (resp., reduced), then S maps to a big Cohen-Macaulay algebra that is a domain (resp., reduced).*

Proof. To start, notice that any element of S killed by a parameter of R will be in the kernel of any map to a big Cohen-Macaulay R-algebra B, so that the map S to B factors through the quotient of S modulo the ideal of elements killed by a parameter of R. When S is a domain, this ideal is the zero ideal, and when S is reduced, this ideal is radical. Hence the quotient is still a domain (resp., reduced). Without loss of generality, we may then assume that no element of S is killed by a parameter of

R. Thus, any nontrivial relation $x_{k+1}s = x_1s_1 + \cdots + x_ks_k$ in S on part of a system of parameters $x_1, \cdots, x_{k+1}$ from R will have $k \geq 1$.

If $k = 1$, then an algebra modification with respect to that relation factors through the $\mathfrak{A}$-transform $\Theta(x_1, x_2; S)$. If $k \geq 2$, then we can factor any algebra modification through an enhanced algebra modification. Therefore, we can map S to a big Cohen-Macaulay R-algebra that is constructed as a very large direct limit of sequences of enhanced algebra modifications and $\mathfrak{A}$-transforms.

Since we are starting with a domain (resp., reduced ring) S and since $\mathfrak{A}$-transforms of domains (resp., reduced rings) are domains (resp., reduced), Lemma 6.5.4 implies that all enhanced algebra modifications and $\mathfrak{A}$-transforms in any sequence will continue to be domains (resp., reduced). Hence, S maps to a big Cohen-Macaulay algebra B that is a direct limit of domains (resp., reduced rings), and so B itself is a domain (resp., reduced). $\square$

We are now able to verify in positive characteristic that every seed can be mapped to a big Cohen-Macaulay algebra that is a domain.

Corollary 6.5.7. *Let R be a local Noetherian ring of positive characteristic. If S is a seed, then S maps to a big Cohen-Macaulay algebra domain.*

Proof. By Proposition 6.3.4, S maps to a minimal seed, and Proposition 6.5.2 implies that the minimal seed is a domain. The previous lemma then implies that a minimal seed can be mapped to a big Cohen-Macaulay algebra that is a domain. $\square$

As a consequence of this result, we can also show that all seeds map to big Cohen-Macaulay algebras with a host of nice properties. We will use an uncountable limit ordinal number in the proof and refer the reader to [HJ, Chapters 7 and 8] for the definitions and properties of such ordinal numbers.

Proposition 6.5.8. *Let (R, m) be a local Noetherian ring of positive characteristic. If S is a seed, then S maps to an absolutely integrally closed, m-adically separated, quasilocal big Cohen-Macaulay algebra domain B.*

Proof. We will construct B as a direct limit of seeds indexed by an uncountable ordinal number. Let β be an uncountable initial ordinal of cardinality $\aleph_1$. Using transfinite induction, we will define an S-algebra S_α, for each ordinal number $\alpha < \beta$, and then we will define B to be the direct limit of all such S_α.

Let $S_0 = S$. Given a seed S_α, we can form a sequence

$$S_\alpha \to S_\alpha^{(1)} \to S_\alpha^{(2)} \to S_\alpha^{(3)} \to S_\alpha^{(4)} =: S_{\alpha+1},$$

where $S_\alpha^{(1)}$ is a minimal seed (and so a domain by Proposition 6.5.2), $S_\alpha^{(2)} = (S_\alpha^{(1)})^+$ (an integral extension of a seed and so a seed by Theorem 6.4.8), $S_\alpha^{(3)}$ is a quasilocal big Cohen-Macaulay R-algebra (which $S_\alpha^{(2)}$ maps to by Lemma 6.1.4), and $S_\alpha^{(4)}$ is the m-adic completion of $S_\alpha^{(3)}$ (which is m-adically separated and a big Cohen-Macaulay algebra by [Bar-Str, Theorem 1.7]). If α is a limit ordinal, then we will define

$$S_\alpha := \varinjlim_{\gamma<\alpha} S_\gamma.$$

Given our definition for S_α, for each ordinal $\alpha < \beta$, we define

$$B := S_\beta = \varinjlim_{\alpha<\beta} S_\alpha.$$

Since each S_α maps to a domain $S_\alpha^{(1)}$, and conversely, each $S_\alpha^{(1)}$ maps to $S_{\alpha+1}$, the ring B can be written as a direct limit of domains and is, therefore, also a domain. Similarly, B is also a direct limit of absolutely integrally closed domains (using $S_\alpha^{(2)}$, for each $\alpha < \beta$). If we let L be the algebraic closure of the fraction field of B and let K_α be the algebraic closure of the fraction field of $S_\alpha^{(2)}$, for each $\alpha < \beta$, then an element $u \in L$ satisfying a monic polynomial equation over B is the image of an element v in some K_α that satisfies a monic polynomial equation over $S_\alpha^{(2)}$, for some α. Since $S_\alpha^{(2)}$ is absolutely integrally closed, v is in $S_\alpha^{(2)}$, and so its image u in L is also in B. Therefore, B is absolutely integrally closed.

Using the rings $S_\alpha^{(3)}$, we can see that B is the direct limit of quasilocal big Cohen-Macaulay R-algebras, and so B is itself a quasilocal big Cohen-Macaulay algebra. Indeed, it is easy to see that $(x_1, \ldots, x_k)B : Bx_{k+1} \subseteq (x_1, \ldots x_k)B$, for each partial system of parameters $x_1, \ldots, x_{k+1}$ of R as this fact is true in each $S_\alpha^{(3)}$. Furthermore, $mB \neq B$, as the opposite would imply that $mS_\alpha^{(3)} = S_\alpha^{(3)}$, for some α, which is impossible in a big Cohen-Macaulay algebra. It is also straightforward to verify that a direct limit of quasilocal rings is quasilocal.

Finally, to see that B is m-adically separated, we note that B is a direct limit of the $S_\alpha^{(4)}$, where each of these rings is m-adically separated. Suppose that $u \in \bigcap_k m^k B$. Then for each k, there exists an ordinal $\alpha(k)$ such that $u \in m^k S_{\alpha(k)}^{(4)}$. Let α be the union of all the $\alpha(k)$. Since $\alpha(k) < \beta$, for all k, we have a countable set of ordinal numbers, and since β is uncountable, we see that $\alpha < \beta$. Therefore, $u \in \bigcap_k m^k S_\alpha$, and so $u \in \bigcap_k m^k S_\alpha^{(4)} = 0$. $\square$

6.6 Tensor Products and Base Change

In this section, let R and S be complete local domains of positive characteristic, and let S be an R-algebra. We will look at two previously open questions about big Cohen-Macaulay algebras and use our previous results to provide positive answers to the questions, which show that the class of seeds over a complete local domain possesses further properties of the class of solid algebras.

First, given two big Cohen-Macaulay R-algebras B and B', does there exist a big

Cohen-Macaulay algebra C such that

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & B' \end{array}$$

commutes? An equivalent question is whether the tensor product of two seeds over R is also a seed over R.

Another open question involves base change $R \to S$ between complete local domains. Given a big Cohen-Macaulay R-algebra B, can B be mapped to a big Cohen-Macaulay S-algebra C such that the diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

commutes? Equivalently, we could ask whether the property of being a seed is preserved under this manner of base change since $B \otimes_R S$ fills in the diagram and would map further to a big Cohen-Macaulay S-algebra if it were a seed over S.

We will show that both of these questions have positive answers in positive characteristic. First, we address the question of why the tensor product of seeds is a seed. We will derive our result from the case of a regular local base ring and make use of tight closure and test elements for the general case. We will need the next two lemmas to obtain our result in the regular case.

Lemma 6.6.1. *If S and T are any commutative rings such that T is flat over S, I is an ideal of S, and $x \in S$, then $IT :_T x = (I :_S x)T$.*

Proof. Over S, we have an exact sequence

$$0 \to \frac{I :_S x}{I} \to \frac{S}{I} \xrightarrow{x} \frac{S}{I}.$$

Since T is flat over S, the sequence remains exact when we tensor with T. Therefore, $IT :_T x/IT \cong (I :_S x/I) \otimes_S T \cong (I :_S x)T/IT$, and we are done. □

Lemma 6.6.2. *If C is a big Cohen-Macaulay algebra over a local ring (S, n) and D is faithfully flat over C, then D is a big Cohen-Macaulay S-algebra.*

Proof. Let $x_1, \ldots, x_{k+1}$ be part of a system of parameters for S, and let d be an element of $(x_1, \ldots, x_k)D :_D x_{k+1}$. Using the lemma above,

$$d \in ((x_1, \ldots, x_k)C :_C x_{k+1})D \subseteq ((x_1, \ldots, x_k)C)D = (x_1, \ldots, x_k)D$$

because C is a big Cohen-Macaulay S-algebra. Finally, as D is faithfully flat over C, and $nC \neq C$, $nD \neq D$ either. □

Now, if A is a regular local ring, and B and B' are big Cohen-Macaulay A-algebras, then B is faithfully flat over A by Proposition 2.3.2. Therefore, $B \otimes_A B'$ is faithfully flat over B', and since B' is a big Cohen-Macaulay A-algebra, we can use the previous lemma to conclude:

Lemma 6.6.3. *If A is a regular local Noetherian ring, and B and B' are big Cohen-Macaulay A-algebras, then $B \otimes_A B'$ is a big Cohen-Macaulay A-algebra as well. Consequently, if S and S' are seeds over A, then $S \otimes_A S'$ is a seed over A.*

We can now establish our first result, concerning tensor products of seeds.

Theorem 6.6.4. *Let (R, m) be a complete local domain of positive characteristic. If $(S_i)_{i \in I}$ is an arbitrary family of seeds over R, then $\bigotimes_{i \in I} S_i$ is also a seed over R. Consequently, if B and B' are big Cohen-Macaulay R-algebras, then there exists a big Cohen-Macaulay R-algebra C filling the commutative diagram:*

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$

Proof. Since a direct limit of seeds over R is a seed over R by Lemma 6.1.2, we may assume that I is a finite set. By induction, we may assume that I consists of two elements. We may then also assume that $S_1 = B$ and $S_2 = B'$ are big Cohen-Macaulay R-algebras. By the Cohen Structure Theorem, R is a module-finite extension of a complete regular local ring A.

We next want to reduce to the case that R is a separable extension of A. Notice that for any $q = p^e$, $R[A^{1/q}]$ is still a module-finite extension of $A^{1/q}$, a complete regular local ring. Furthermore, $B[A^{1/q}]$ and $B'[A^{1/q}]$ are clearly still big Cohen-Macaulay $R[A^{1/q}]$-algebras, and if $B[A^{1/q}] \otimes_{R[A^{1/q}]} B'[A^{1/q}]$ is a seed over $R[A^{1/q}]$, then it is also a seed over R, which will show that $B \otimes_R B'$ is a seed over R, as needed. We may therefore replace A and R by $A^{1/q}$ and $R[A^{1/q}]$ for any $q \geq 1$. We claim that for any $q \gg 1$, we obtain a separable extension $A^{1/q} \to R[A^{1/q}]$. Indeed, suppose that R is not separable over A. Passing to the fraction field K of A, we have that $K \otimes_A R$ is a finite product of finite field extensions of K. We may then assume that L is a finite inseparable field extension of K and show that $L[K^{1/q}]$ is separable over $K^{1/q}$, for some $q \gg 1$. For any element y of L whose minimal polynomial is inseparable, we can find q sufficiently large so that the minimal polynomial of y in $L[K^{1/q}]$ over $K^{1/q}$ becomes separable. Since L is a finite extension, for any q sufficiently large, $L[K^{1/q}]$ becomes separable over $K^{1/q}$, which implies that $R[A^{1/q}]$ will be separable over $A^{1/q}$.

We can now assume without loss of generality that R is separable over A. Let J be the ideal of $R \otimes_A R$ generated by all elements killed by an element of A. We claim that $R_0 := (R \otimes_A R)/J$ is a reduced ring. As R_0 is a separable extension of A (since R is separable), if r is a nilpotent element, then r satisfies a polynomial X^q over A. Thus, the separability implies that r must also satisfy the polynomial X, i.e., $r = 0$.

Since B and B' are big Cohen-Macaulay R-algebras and R is a module-finite extension of A, we also have that B and B' are big Cohen-Macaulay A-algebras. By Lemma 6.6.3, $B \otimes_A B'$ is also a big Cohen-Macaulay A-algebra. The R-algebra structures of B and B' induce a natural map from $R \otimes_A R$ to $B \otimes_A B'$. Since the latter ring is big Cohen-Macaulay over A, the ideal J in $R \otimes_A R$ is contained in the kernel of the map to $B \otimes_A B'$. Therefore, the map $R \otimes_A R \to B \otimes_A B'$ factors through R_0.

Since R is a domain and a homomorphic image of $R \otimes_A R$ (and so a homomorphic image of R_0), the kernel of $R_0 \to R$ is a prime ideal P. Moreover, since R_0 is also a module-finite extension of A, the dimensions of R and R_0 are the same, so that P is a minimal prime of R_0. Finally, notice that the kernel of $B \otimes_A B' \to B \otimes_R B'$ is the extended ideal $P(B \otimes_A B')$, since it is also generated by the elements $r \otimes 1 - 1 \otimes r$, for all $r \in R$. We therefore obtain a commutative diagram

$$\begin{tikzcd} B \otimes_A B' \arrow[r] & B \otimes_R B' \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

where the horizontal maps are the result of killing the minimal prime ideal P of R_0 (resp., $P(B \otimes_A B')$).

We will now show that $B \otimes_R B'$ is a seed over R by constructing a weak durable colon-killer with respect to the systems of parameters of R that are contained in A. We can then apply Proposition 6.2.4 and Theorem 6.2.8 to finish the proof.

Since R_0 is reduced and P is a minimal prime, there exists $c' \notin P$ such that $c'P = 0$. Therefore, $c'P(B \otimes_A B') = 0$ too. As $c' \notin P$, its image in R is nonzero. Since R is a complete local domain (and so local, reduced, and excellent), R has a test element $c'' \neq 0$ by Theorem 2.2.6. Let $c = c'c''$, a nonzero test element of R, and let d be a lifting of c to R_0 so that $dP(B \otimes_A B') = 0$.

Now, suppose that $x_1, \ldots, x_n$ is a system of parameters of A, and suppose that $x_{k+1}u \in (x_1, \ldots, x_k)(B \otimes_R B')$, for some $k \leq n-1$. Then

$$x_{k+1}u \in ((x_1, \ldots, x_k) + P)(B \otimes_A B'),$$

and $x_{k+1}du \in (x_1, \ldots, x_k)(B \otimes_A B')$. Since $B \otimes_A B'$ is a big Cohen-Macaulay A-algebra, $du \in (x_1, \ldots, x_k)(B \otimes_A B')$, and so $cu \in (x_1, \ldots, x_k)(B \otimes_R B')$, as c and d have the same image in $B \otimes_R B'$. Therefore, c is a colon-killer for systems of parameters of A in $B \otimes_R B'$. By Proposition 6.2.4, we may replace c by a power that is a colon-killer for systems of parameters of R.

Finally, suppose that $c^N \in \bigcap_k m^k(B \otimes_R B')$. Then Theorem 2.5.4 and Theorem 2.5.5 imply that $c^N \in \bigcap_k ((m^k)^*)$ since $B \otimes_R B'$ is solid over R. (The solidity of $B \otimes_R B'$ follows from Proposition 2.5.2(a).) As c is a test element in R, $c^{N+1} \in \bigcap_k m^k = 0$, a contradiction. Hence, c is a durable colon-killer in $B \otimes_R B'$, and so $B \otimes_R B'$ is a seed over R by Theorem 6.2.8. □

We now proceed to the question of whether being a seed over a complete local domain of positive characteristic is a property that is preserved by base change to another complete local domain. If $R \to S$ is a map of complete local rings, then [AFH, Theorem 1.1] says that the map factors through a complete local ring R' such that $R \to R'$ is faithfully flat with regular closed fiber, and $R' \to S$ is surjective. It therefore suffices to treat the cases of a flat local map with regular closed fiber and the case of a surjective map with kernel a prime ideal. We start with an elementary lemma and then prove the result for the flat local case.

Lemma 6.6.5. *Let (A, m) be a local Noetherian ring, and let B be a flat, local Noetherian R-algebra. If $y_1, \ldots, y_t$ is a regular sequence on B/mB, then $y_1, \ldots, y_t$ is* *a regular sequence on B, and $B/(y_1,\dots,y_t)B$ is flat over A.*

Proof. The proof is immediate by induction on t, where the base case of $t = 1$ is given by [Mat, (20.F)]. $\square$

With this lemma, we are ready to prove our base change result for flat local maps. It is perhaps interesting to note that our argument only requires that the closed fiber is Cohen-Macaulay, not regular. Unlike the surjective case, we will not need to assume that our rings have positive characteristic, are complete, or are domains.

Proposition 6.6.6. *Let $R \to S$ be a flat local map of Noetherian local rings with a Cohen-Macaulay closed fiber S/mS, where m is the maximal ideal of R. If T is a seed over R, then $T \otimes_R S$ is a seed over S.*

Proof. It suffices to assume that $T = B$ is a big Cohen-Macaulay R-algebra, and by [Bar-Str, Theorem 1.7], it suffices to show that a single system of parameters of S is a regular sequence on $B \otimes_R S$ since this result shows that such a ring maps to an S-algebra where every system of parameters of S is a regular sequence.

Fix a system of parameters $x_1,\dots,x_d$ for R. Since S is faithfully flat over R, we have the dimension equality $\dim S = \dim R + \dim S/mS$. Hence, the images of $x_1,\dots,x_d$ in S can be extended to a full system of parameters $x_1,\dots,x_d,x_{d+1},\dots,x_n$ of S, where $x_{d+1},\dots,x_n$ is a system of parameters for S/mS. As $x_1,\dots,x_d$ form a regular sequence on B, for any $1 \le k \le d-1$, we have an exact sequence:

$$0 \to B/(x_1,\dots,x_k)B \xrightarrow{x_{k+1}} B/(x_1,\dots,x_k)B,$$

and since S is flat over R, the sequence remains exact after tensoring with S. Thus, x_{k+1} is not a zerodivisor on

$$(B/(x_1,\dots,x_k)B) \otimes_R S \cong (B \otimes_R S)/(x_1,\dots,x_k)(B \otimes_R S),$$

for any $1 \leq k \leq d-1$.

It now suffices to show that $x_{d+1}, \ldots, x_n$ is a regular sequence on the quotient

$$\overline{B} := (B \otimes_R S)/(x_1, \ldots, x_k)(B \otimes_R S).$$

Let $I := (x_1, \ldots, x_d)R$, and let $\overline{R} := R/I$, and $\overline{S} := S/IS$. Then $\overline{S}$ is faithfully flat over $\overline{R}$, and since $x_{d+1}, \ldots, x_n$ is a system of parameters for the Cohen-Macaulay ring S/mS, it is a regular sequence on $S/mS \cong \overline{S}/\overline{m}\overline{S}$, where $\overline{m} = m/I$, the maximal ideal of $\overline{R}$. We can now apply Lemma 6.6.5 to $\overline{R}$ and $\overline{S}$ to conclude that $x_{d+1}, \ldots, x_n$ is a regular sequence on $\overline{S}$ and that $\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}$ is flat over $\overline{R}$, for all $d+1 \leq k \leq n$.

For any $d \leq k \leq n-1$, we have a short exact sequence

$$0 \to \overline{S}/(x_{d+1}, \ldots, x_k)\overline{S} \xrightarrow{x_{k+1}} \overline{S}/(x_{d+1}, \ldots, x_k)\overline{S} \to \overline{S}/(x_{d+1}, \ldots, x_k, x_{k+1})\overline{S} \to 0,$$

where $x_{d+1}, \ldots, x_k$ is the empty sequence when $k = d$. Since $\overline{S}/(x_{d+1}, \ldots, x_{k+1})\overline{S}$ is flat over $\overline{R}$, we have $\mathrm{Tor}_1^{\overline{R}}(\overline{B}, \overline{S}/(x_{d+1}, \ldots, x_{k+1})\overline{S}) = 0$. Therefore,

$$0 \to \overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}) \xrightarrow{x_{k+1}} \overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S})$$

is exact, and since

$$\overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}) \cong (\overline{B} \otimes_{\overline{R}} \overline{S})/(x_{d+1}, \ldots, x_k)(\overline{B} \otimes_{\overline{R}} \overline{S}),$$

$x_{d+1}, \ldots, x_n$ is a possibly improper regular sequence on $\overline{B} \otimes_{\overline{R}} \overline{S}$. We can now finally see that $x_1, \ldots, x_d, x_{d+1}, \ldots, x_n$ is a possibly improper regular sequence on $B \otimes_R S$ as $\overline{B} \otimes_{\overline{R}} \overline{S} \cong (B \otimes_R S)/I(B \otimes_R S)$. If, however, $(B \otimes_R S)/n(B \otimes_R S) = 0$, where n is the maximal ideal of S, then $B \otimes_R (S/nS) = 0$, which implies that the product $(B/mB) \otimes_{R/m} (S/nS) = 0$ over the field R/m. Therefore, $B/mB = 0$ or $S/nS = 0$, but neither of these occurs, so we have a contradiction. Hence, $x_1, \ldots, x_n$ is a regular sequence on $B \otimes_R S$, and so $B \otimes_R S$ is a seed. $\square$

We will now deal with the case of a surjective map $R \to S$ of complete local domains of positive characteristic. We can immediately reduce to the case where $S = R/P$, where P is a height 1 prime ideal of R. We will first demonstrate the result when R is normal and then show how the problem can be reduced to the normal case using Theorem 6.4.8 concerning integral extensions of seeds.

In the normal case, we will make use of test elements again. When R is normal, the singular locus I has height at least 2. Since R_c is regular for any $c \in I$, when R is also a reduced excellent local ring, [HH6, Theorem 6.1] implies that some power c^N is a test element of R. Hence, if P is a height 1 prime of R, there exists a test element c of R not in P. We record this fact in the following lemma.

Lemma 6.6.7. *Let R be a normal excellent local ring of positive characteristic. If P is a height 1 prime of R, then there is a test element $c \in R \setminus P$.*

Lemma 6.6.8. *Let (R, m) be a complete local, normal domain of positive characteristic, and let $S = R/P$, where P is a height 1 prime of R. If T is a seed over R, then T/PT is a seed over S.*

Proof. It suffices to assume that $B = T$ is a big Cohen-Macaulay R-algebra. Since R is normal and ht $P = 1$, we see that R_P is a DVR. Therefore, PR_P is a principal ideal, which we may assume is generated by the image of an element $x \in R$. Each element of P is then multiplied into xR by some element of $R \setminus P$, and since P is finitely generated, there exists $c' \in R \setminus P$ such that $c'P \subseteq xR$. By Lemma 6.6.7, there exists a test element $c'' \in R \setminus P$, and so if we put $c := c'c''$, then c is a test element, $cP \subseteq xR$, and c is not in P.

We claim that c is a weak durable colon-killer for B/PB so that B/PB will be a seed over $S = R/P$ by Theorem 6.2.8. Extend x to a full system of parameters

$x, x_2, \ldots, x_d$ for R. Then $x_2^t, \ldots, x_d^t$ is a system of parameters for S, for any $t \in \mathbb{N}$. Suppose that $bx_{k+1}^t \in (x_2^t, \ldots, x_k^t)B/PB$, for some $k \leq d-1$ and some t. This relation lifts to a relation $bx_{k+1}^t \in (x_2^t, \ldots, x_k^t)B + PB$ in B, and so $cbx_{k+1}^t \in (x, x_2^t, \ldots, x_k^t)B$ because c multiplies P into xR. Since B is a big Cohen-Macaulay R-algebra, we have $cb \in (x, x_2^t, \ldots, x_k^t)B$, and so $cb \in (x_2^t, \ldots, x_k^t)B/PB$.

To finish, suppose that $c^N \in \bigcap_k (m/P)^k B/PB$, where m/P is the maximal ideal of $S = R/P$. We can then lift to B to obtain $c^N \in \bigcap_k (m^k + P)B$. Since R is a complete local domain, and B is a big Cohen-Macaulay R-algebra, Theorem 2.5.6 implies that $c^N \in \bigcap_k (m^k + P)^*$, and since c was chosen to be a test element, we have $c^{N+1} \in \bigcap_k (m^k + P) = P$. We now have a contradiction as $c \notin P$. Therefore, the image of c in B/PB is a weak durable colon-killer, and so B/PB is a seed over S by Theorem 6.2.8. $\square$

We finally treat the case of an arbitrary surjection of complete local domains by reducing to the case of the previous lemma.

Proposition 6.6.9. *Let $R \to S$ be a surjective map of positive characteristic complete local domains. If T is a seed over R, then $T \otimes_R S$ is a seed over S.*

Proof. We can immediately assume that the kernel of $R \to S$ is a height 1 prime P of R. Let R' be the normalization of R in its fraction field. Then R' is also a complete local domain. Since R' is an integral extension of R, there exists a height 1 prime Q lying over P.

By Corollary 6.5.7, T maps to a big Cohen-Macaulay R-algebra domain B, and so we may replace T by B and assume that T is a domain. We then have an integral extension $T[R']$ of T inside the fraction field of T. Since T is a seed over R, Theorem 6.4.8 implies that $T[R']$ is also a seed over R. Therefore, $T[R']$ maps to a big Cohen-

Macaulay R-algebra C (which is also an R'-algebra), and so Corollary 6.2.6 implies that C is a big Cohen-Macaulay R'-algebra since R' is integral over R. We now have the commutative diagram:

$$\begin{tikzcd}
& C \arrow[rr] & & C/QC \\
T \arrow[ur] \arrow[rr] & & T/PT \arrow[ur] & \\
& R' \arrow[uu] \arrow[rr] & & R'/Q \arrow[uu] \\
R \arrow[uu] \arrow[ur] \arrow[rr] & & S \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

since $S = R/P$ and Q lies over P. By Lemma 6.6.8, C/QC is a seed over R'/Q. Since R'/Q is an integral extension of $S = R/P$, every system of parameters of S is a system of parameters of R'/Q, and so C/QC is also a seed over S, but this implies that T/PT is also a seed over S, as needed. $\square$

Now that we have shown that the property of being a seed over a complete local domain is preserved by flat base change with a regular closed fiber (Proposition 6.6.6) and by surjections (Proposition 6.6.9), we may apply [AFH, Theorem 1.1] to factor any map of complete local domains into these two maps. We therefore arrive at the following theorem, which answers the base change question asked at the beginning of the section.

Theorem 6.6.10. *Let $R \to S$ be a local map of positive characteristic complete local domains. If T is a seed over R, then $T \otimes_R S$ is a seed over S. Consequently, if B is a big Cohen-Macaulay R-algebra, then there exists a big Cohen-Macaulay S-algebra*

C filling the commutative diagram:

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

6.7 Seeds and Tight Closure in Positive Characteristic

Because we have the two main results from the last section (Theorems 6.6.4 and 6.6.10), we can now use the class of all big Cohen-Macaulay R-algebras $\mathscr{B}(R)$, where R is a complete local domain of characteristic p, to define a closure operation for all Noetherian rings of positive characteristic. A key point is that $\mathscr{B}(R)$ is a directed family and has certain base change properties. By Theorem 2.5.6, our new closure operation is equivalent to tight closure for complete local domains of positive characteristic, but the results above imply many of the properties one would want in a good closure operation directly from the properties of big Cohen-Macaulay algebras, independent of tight closure. This result adds evidence to the idea that such a closure operation can be defined for more general classes of rings.

Definition 6.7.1. Let R be a complete local domain of positive characteristic, and let $N \subseteq M$ be finitely generated R-modules. Let $N_M^\natural$ be the set of all elements $u \in M$ such that $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$, for some big Cohen-Macaulay R-algebra B.

Let S be a Noetherian ring of positive characteristic, and let $N \subseteq M$ be finitely generated S-modules. Let $N_M^\natural$ be the set of all $u \in M$ such that for all S-algebras T, where T is a complete local domain, $1 \otimes u \in \operatorname{Im}(T \otimes_S N \to T \otimes_S M)^\natural_{T \otimes_S M}$. We will call $N_M^\natural$ the $\natural$-*closure* of N in M.

We will see in Lemma 7.4.4 that the two definitions of $\natural$-closure coincide for complete local domains. We also show in the following chapter that $\natural$-closure satisfies many nice properties, including persistence, $(IS)^{\natural}_{S} \cap R \subseteq I^{\natural}_{R}$ for module-finite extensions, $I^{\natural} = I$ for ideals in a regular ring, phantom acyclicity, and colon-capturing.

CHAPTER 7

Axioms for a Good Closure Operation

In equal characteristic, tight closure, or similar methods, can be used to show the weakly functorial existence of big Cohen-Macaulay algebras. See Theorem 2.3.5, [HH3], [HH7, Section 3], and [Ho3]. In these cases, the notion of weakly functorial is that given in Theorem 2.3.5.

By *weakly functorial* existence of big Cohen-Macaulay algebras for a class of local Noetherian rings $\mathscr{C}$, we will mean, however, that we can assign a *directed family* $\mathscr{B}(R)$ of big Cohen-Macaulay R-algebras to each $R \in \mathscr{C}$ such that given an R-algebra S, with $S \in \mathscr{C}$, and $B \in \mathscr{B}(R)$, there exists $C \in \mathscr{B}(S)$ such that the diagram below commutes.

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

We call $\mathscr{B}(R)$ a *directed family* if for all $B, C \in \mathscr{B}(R)$, there exists $D \in \mathscr{B}(R)$ such that B and C both map to D as R-algebras.

The intent of this chapter is to investigate, for a given class of rings, what may be the minimal necessary axioms which a closure operation must possess in order to be a good analogue of tight closure. Another motivation is that we hope that the existence of a closure operation satisfying sufficiently powerful axioms will imply the weakly functorial existence of big Cohen-Macaulay algebras, although we have not been able to complete this program here.

In the second and third sections, we will show various consequences of the existence of a closure operation that satisfies our list of axioms. The third section concentrates on an analogue of phantom extensions (see Section 5.2.1 or [HH5] for the tight closure notion of phantom extension). Phantom extensions are used in [HH5, Discussion 5.15] to give another proof of the existence of big Cohen-Macaulay modules in positive characteristic. We introduce our analogue of phantom extensions, and the properties we know about them, in the hope that they may eventually be used in a proof that our axioms (or a larger set) imply the existence of big Cohen-Macaulay algebras.

As part of the "minimality" condition on our axioms, we will require that each axiom can be derived from the weakly functorial existence of big Cohen-Macaulay algebras. We will define a closure operation on ideals and modules using contracted-expansion from big Cohen-Macaulay algebras and then show that this closure operation satisfies all of our axioms. We will also extend our definition to general Noetherian rings using the definition for complete local rings.

If $\mathscr{C}$ is the class of complete local domains of positive characteristic p, then R^+ is a big Cohen-Macaulay algebra for all $R \in \mathscr{C}$. Given a map $R \to S$, there is a "compatible" map $R^+ \to S^+$. If $\mathscr{B}(R) = \{R^+\}$ for all $R \in \mathscr{C}$, then the plus closure is the derived closure operation from this family of big Cohen-Macaulay algebras.

In the previous chapter we showed that the family of all big Cohen-Macaulay R-algebras $\mathscr{B}(R)$ in the class of complete local domains of positive characteristic p is a directed family (see Theorem 6.6.4) and induces a weakly functorial existence of big Cohen-Macaulay algebras (see Theorem 6.6.10), so that the family of all big Cohen-Macaulay algebras in positive characteristic very naturally induces a good closure operation, i.e., setting $\mathscr{B}(R)$ to be the class of all big Cohen-Macaulay R-algebras wills give a closure operation that will satisfy our axioms. Over complete local domains, this operation coincides with tight closure.

7.1 The Axioms

Definition 7.1.1. Let $\mathscr{R}$ be a full subcategory of Noetherian rings such that, for all $R \in \mathscr{R}$, all homomorphic images and localizations of R are in $\mathscr{R}$, all completions of localizations are in $\mathscr{R}$, and all R-algebra DVRs are in $\mathscr{R}$. We will call such a category $\mathscr{R}$ a *suitable category* of Noetherian rings. Let $\mathscr{C}$ be the full subcategory of $\mathscr{R}$ of all complete local domains.

A closure operation satisfying the following list of axioms will be denoted by $N^\natural_M$ for the closure of N within M, and throughout we will call this the $\natural$-*closure* of N in M.

Axioms 7.1.2. In the following, R and S are rings in $\mathscr{R}$, a suitable category, with S an R-algebra; I is an ideal of R; and $N \subseteq M$ and W are finitely generated R-modules. Let $\mathscr{C}$ be the full subcategory of $\mathscr{R}$ of all complete local domains.

(1) $N^\natural_M$ is a submodule of M containing N.

(2) If $M \subseteq W$, then $N^\natural_W \subseteq M^\natural_W$.

(3) If $f : M \to W$, then $f(N^\natural_M) \subseteq f(N)^\natural_W$.

(4) (persistence) $\mathrm{Im}(S \otimes_R N^\natural_M \to S \otimes_R M) \subseteq \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}$. In particular, if $N = I$ and $M = R$, we have $I^\natural_R S \subseteq (IS)^\natural_S$.

(5) $I^\natural_R N^\natural_M \subseteq (IN)^\natural_M$.

(6) $(N^\natural_M)^\natural_M = N^\natural_M$, i.e., the $\natural$-closure of N in M is closed in M.

(7) If $N^\natural_M = N$, then $0^\natural_{M/N} = 0$.

(8) If $u + \mathfrak{p}M \in \mathrm{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)^\natural_{M/\mathfrak{p}M}$, calculated over $R/\mathfrak{p}R$, for all minimal

primes $\mathfrak{p}$ of R, then $u \in N_M^\natural$.

(9) If $u/1 \in (N_m)^\natural_{M_m}$, calculated over R_m, for all maximal ideals m of R, then $u \in N_M^\natural$.

(10) Let S be faithfully flat over R. If $1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S \otimes_R M}$, calculated over S, then $u \in N_M^\natural$. In particular, if $N = I$ and $M = R$, we have $(IS)^\natural_S \cap R \subseteq I^\natural_R$.

(11) Let S be a module-finite extension of R. If $1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S \otimes_R M}$, calculated over S, then $u \in N_M^\natural$.

(12) If R is regular, then $N_M^\natural = N$.

(13) (phantom acyclicity) For $R \in \mathscr{C}$, a complete local domain, let $G_\bullet$ be a finite free complex over R:

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0.$$

Let α_i be the matrix map $G_i \to G_{i-1}$, and let r_i be the determinantal rank of α_i, for $1 \leq i \leq n$. Let b_i be the free rank of G_i, for $0 \leq i \leq n$. Denote the ideal generated by the size r minors of a matrix map α by $I_r(\alpha)$. If $b_i = r_i + r_{i-1}$ and $\operatorname{ht} I_{r_i}(\alpha_i) \geq i$, for all $1 \leq i \leq n$, then $Z_i \subseteq (B_i)^\natural_{G_i}$, where B_i is the image of α_{i+1} and Z_i is the kernel of α_i. In other words, if the previous rank and height conditions are satisfied, then the cycles are contained in the $\natural$-closure of the boundaries.

Remark 7.1.3. Axioms (4), (8), (9), and (10) allow us to reduce many problems to the case of complete local domains since the completion of a local ring is faithfully flat over the base. Essentially, this allows us to ignore the larger class $\mathscr{R}$ and mainly work over the class of complete local domains $\mathscr{C}$ when necessary.

Remark 7.1.4. Comparing to Proposition 2.2.3 and Theorems 2.2.9, 2.2.10, and

2.2.11, we see that tight closure is known to possess all of these axioms, except for Axioms (9) and (10). If, however, we defined tight closure for complete local domains and then extended to all positive characteristic rings R by testing tight closure over all complete local R-algebra domains, then we would have the notion of *formal tight closure*, as described by Hochster in [Ho3, Remark 8.7]. Formal tight closure obeys all of the axioms above.

7.2 Corollaries of the Axioms

Using the axioms of the last section, we can derive many nice properties. Included amongst these properties are colon-capturing and the fact that, like tight closure, the $\natural$-closure of an ideal is contained in its integral closure. Throughout this section, $\mathscr{R}$ is a suitable category, and $\mathscr{C}$ is the full subcategory of $\mathscr{R}$ consisting of all complete local domains of $\mathscr{R}$.

Lemma 7.2.1 (colon-capturing)**.** *Let R be a local ring in $\mathscr{R}$ such that every system of parameters of R is a system of parameters in $\widehat{R}/\mathfrak{p}$, for every minimal prime $\mathfrak{p}$ of $\widehat{R}$. If $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, then*

$$(x_1, \ldots, x_k) :_R x_{k+1} \subseteq (x_1, \ldots, x_k)^\natural.$$

Proof. Using Axioms (8) and (10) and our hypothesis on R, we can assume R is a complete local domain.

Let $G_\bullet := K_\bullet(x_1, \ldots, x_{k+1}; R)$, the Koszul complex for $x_1, \ldots, x_{k+1}$. Let Z be the kernel of $G_1 \to G_0$, and let B be the image of $G_2 \to G_1$. Let $\epsilon_1, \ldots, \epsilon_{k+1}$ be the standard basis for $G_1 = K_1(x_1, \ldots, x_{k+1}; R) \cong R^{k+1}$. Then Z is the collection of vectors $(r_1, \ldots, r_{k+1})$ such that $r_1 x_1 + \cdots + r_{k+1} x_{k+1} = 0$ in R, and B is the submodule of G_1 generated by $\{x_j \epsilon_i - x_i \epsilon_j \,|\, 1 \le i, j \le k+1\}$. Let $\pi : G_1 \twoheadrightarrow R$ be the projection map $(r_1, \ldots, r_{k+1}) \mapsto r_{k+1}$, so that $\pi(B) = (x_1, \ldots, x_k)R$. Axiom (3) implies that $\pi(B_{G_1}^\natural) \subseteq \pi(B)_R^\natural = (x_1, \ldots, x_k)^\natural$. If $ux_{k+1} = x_1r_1 + \cdots + x_kr_k$, then the vector $(-r_1, \ldots, -r_k, u)$ is in Z. By phantom acyclicity, $(-r_1, \ldots, -r_k, u) \in B_{G_1}^\natural$, which implies that $u = \pi(-r_1, \ldots, -r_k, u) \in (x_1, \ldots, x_k)^\natural$. $\square$

Lemma 7.2.2. *If $R \in \mathscr{R}$ and $I \subseteq R$, then $I^\natural \subseteq \overline{I}$, the integral closure of I.*

Proof. An element $x \in R$ is in $\overline{I}$ if and only if $h(x) \in IV$ for every homomorphism $h : R \to V$ such that V is a DVR, and $\ker h$ is a minimal prime of R. (See [HH1, Section 5] or [L].) By Axiom (4), $x \in I^\natural$ implies that $h(x) \in (IV)_V^\natural$, for all such maps $h : R \to V$. Axiom (12), however, implies that $(IV)_V^\natural = IV$ so that the conclusion holds. $\square$

The following list of properties mimics the list of properties given for tight closure of modules in [HH1, Section 8].

Lemma 7.2.3. *Let $R \in \mathscr{R}$. In the following, N, N', N_i, and M_i are all R-submodules of the finitely generated R-module M, and I is an ideal of R.*

(a) *Let $\mathcal{I}$ be any set. If $N_i \subseteq M$, for all $i \in \mathcal{I}$, then $(\bigcap_{i\in\mathcal{I}} N_i)_M^\natural \subseteq \bigcap_{i\in\mathcal{I}}(N_i)_M^\natural$.*

(b) *Let $\mathcal{I}$ be any set. If N_i is $\natural$-closed in M, for all $i \in \mathcal{I}$, then $\bigcap_{i\in\mathcal{I}} N_i$ is $\natural$-closed in M.*

(c) $(N_1 + N_2)_M^\natural = ((N_1)_M^\natural + (N_2)_M^\natural)_M^\natural$.

(d) $(IN)_M^\natural = (I_R^\natural N_M^\natural)_M^\natural$.

(e) *If N is $\natural$-closed in M, then $N :_M I$ is $\natural$-closed in M, and $N :_R N'$ is $\natural$-closed in R.*

(f) *Let $N' \subseteq N \subseteq M$. Then $u \in N_M^\natural$ if and only if $u + N' \in (N/N')_{M/N'}^\natural$.*

(g) *If $\mathcal{I}$ is a finite set, $N = \bigoplus_{i\in\mathcal{I}} N_i$, and $M = \bigoplus_{i\in\mathcal{I}} M_i$, then $N_M^\natural = \bigoplus_{i\in\mathcal{I}}(N_i)_M^\natural$.*

(h) *Let J be the nilradical of R. Then $0_R^\natural = J$. If R is reduced, then $0_R^\natural = 0$.*

(i) *Let J be the nilradical of R. Then $JM \subseteq N_M^\natural$.*

Proof. (a) Let $u \in (\bigcap_{i \in \mathcal{I}} N_i)^\natural_M$. By Axiom (2), $u \in (N_i)^\natural_M$ for all $i \in \mathcal{I}$.

(b) Since each N_i is $\natural$-closed, the conclusion follows from (a) and Axiom (6).

(c) As $N_i \subseteq (N_i)^\natural_M$, Axiom (2) implies that

$$(N_1 + N_2)^\natural_M \subseteq ((N_1)^\natural_M + (N_2)^\natural_M)^\natural_M.$$

Conversely, $N_i \subseteq N_1 + N_2$, with Axiom (2), implies that $(N_i)^\natural_M \subseteq (N_1 + N_2)^\natural_M$. Therefore, $(N_1)^\natural_M + (N_2)^\natural_M \subseteq (N_1 + N_2)^\natural_M$. Axioms (2) and (6) then yield

$$((N_1)^\natural_M + (N_2)^\natural_M)^\natural_M \subseteq ((N_1 + N_2)^\natural_M)^\natural_M = (N_1 + N_2)^\natural_M.$$

(d) $IN \subseteq I^\natural_R N^\natural_M$ implies that $(IN)^\natural_M \subseteq (I^\natural_R N^\natural_M)^\natural_M$ by Axiom (2). For the converse, Axiom (6) implies that it is sufficient to show $I^\natural_R N^\natural_M \subseteq (IN)^\natural_M$, which is exactly Axiom (5).

(e) Since $I(N :_M I) \subseteq N$ (resp., $(N :_R N')N' \subseteq N$), by Axioms (2) and (5),

$$I^\natural_R (N :_M I)^\natural_M \subseteq (I(N :_M I))^\natural_M \subseteq N^\natural_M = N$$

(resp., $(N :_R N')^\natural_R (N')^\natural_M \subseteq ((N :_R N')N')^\natural_M \subseteq N^\natural_M = N$). Therefore,

$$(N :_M I)^\natural_M \subseteq N :_M I^\natural_R$$

(resp., $(N :_R N')^\natural_R \subseteq N :_R (N')^\natural_M$). Since $N :_M I^\natural_R$ is trivially contained in $N :_M I$ (resp., $N :_R (N')^\natural_M \subseteq N :_R N'$), $N :_M I$ (resp., $N :_R N'$) is $\natural$-closed.

(f) Since $(M/N')/(N/N') \cong M/N$, it is enough to show the case where $N' = 0$. Let $\pi : M \to M/N$ be the natural surjection. If $u \in N^\natural_M$, then Axiom (3) implies that $u + N = \pi(u) \in (\pi(N))^\natural_{M/N} = 0^\natural_{M/N}$. Conversely, if $u + N \in 0^\natural_{M/N}$, then applying Axiom (3) to the map $M/N \to M/N^\natural_M$ yields $u + N^\natural_M \in 0^\natural_{M/N^\natural_M}$. By Axiom (7), 0 is $\natural$-closed in $M/N^\natural_M$, so $u + N^\natural_M$ is zero in the quotient.

(g) Let $\pi_i : M \to M_i$ be the natural projection, and let $\iota_i : M_i \to M$ be the natural inclusion, for all i. By Axiom (3), $\pi_i(N_M^\natural) \subseteq \pi_i(N)_{M_i}^\natural = (N_i)_{M_i}^\natural$. Therefore, $N_M^\natural \subseteq \bigoplus_i (N_i)_{M_i}^\natural$ in $M = \bigoplus_i M_i$. Conversely, we apply Axiom (3) to the map ι_i to see $\iota_i((N_i)_{M_i}^\natural) \subseteq \iota_i(N_i)_M^\natural$, which is contained in $N_M^\natural$ by Axiom (2). Thus, $\bigoplus_i (N_i)_{M_i}^\natural \subseteq N_M^\natural$.

(h) First, assume that R is a domain. Then $u \in 0_R^\natural$, and by Lemma 7.2.2, u is in the integral closure of the zero ideal. Since (0) is a prime ideal, it is integrally closed, and $u = 0$. For a general R, if $u \in 0_R^\natural$, then for all minimal primes $\mathfrak{p}$ of R, $\overline{u} \in 0_{R/\mathfrak{p}}^\natural = 0$. This implies that u is in all minimal primes of R, and so u is in J. Conversely, if $u \in J$, then $\overline{u} = 0$ in $R/\mathfrak{p}$ for all minimal primes $\mathfrak{p}$, and $\overline{u} \in 0_{R/\mathfrak{p}}^\natural$ for all such $\mathfrak{p}$. By Axiom (8), u is in $0_R^\natural$.

(i) By (h), $JM = 0_R^\natural M$, and since $M = M_M^\natural$, we have $JM \subseteq (0M)_M^\natural = 0_M^\natural$ by Axiom (5). Then Axiom (2) implies that $JM \subseteq 0_M^\natural \subseteq N_M^\natural$.

□

7.3 $\natural$-Phantom Extensions

In this section we want to define a notion of phantom extensions for $\natural$-closure. We will continue the use of the axioms of Section 7.1 as our definition of $\natural$-closure and will assume that all rings are in a suitable category $\mathscr{R}$.

Since phantom extensions were used in [HH5] to produce a new proof of the existence of big Cohen-Macaulay modules in positive characteristic, a study of $\natural$-phantom extensions may be very useful in showing the existence of big Cohen-Macaulay modules, or even algebras, in suitable categories.

Recall from Section 5.2.1 or [HH5] that for a map $\alpha : N \to M$ of finitely generated R-modules, α is a phantom extension if there exists $c \in R^\circ$ such that for all $e \gg 0$, there exists $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$.

Via the Yoneda correspondence, every short exact sequence

$$0 \to N \xrightarrow{\alpha} M \to Q \to 0$$

corresponds to a unique element ϵ of $\mathrm{Ext}^1_R(Q, N)$. Let $P_\bullet$ be a projective resolution of $Q = M/\alpha(N)$. Then $\mathrm{Ext}^1_R(Q, N)$ is isomorphic to $H^1(\mathrm{Hom}_R(P_\bullet, N))$, and ϵ corresponds to a unique element of $H^1(\mathrm{Hom}_R(P_\bullet, N))$.

Given the map $\alpha : N \to M$, and the corresponding element ϵ of $\mathrm{Ext}^1_R(Q, N)$, Hochster and Huneke called ϵ *phantom* if a cocycle representative of ϵ in $\mathrm{Hom}_R(P_1, N)$ is in the tight closure of $\mathrm{Im}(\mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N))$ within $\mathrm{Hom}_R(P_1, N)$. In the case that $N = R$, Hochster and Huneke provide the following equivalence.

Theorem 7.3.1 (Theorem 5.13, [HH5]). *Let R be a reduced Noetherian ring of positive characteristic. An exact sequence*

$$0 \to R \xrightarrow{\alpha} M \to Q \to 0$$

is a phantom extension if and only if the corresponding element $\epsilon \in \mathrm{Ext}^1_R(Q, R)$ is phantom in the sense described just above.

We will use this latter property to define $\natural$-*phantom extensions* with respect to $\natural$-closure.

Definition 7.3.2. Let R be a reduced Noetherian ring, M be a finitely generated R-module, and $\alpha : R \to M$ an injective R-linear map. With $Q = M/\alpha(R)$, we have an induced short exact sequence

$$0 \to R \xrightarrow{\alpha} M \to Q \to 0.$$

Let $\epsilon \in \mathrm{Ext}^1_R(Q, R)$ be the element corresponding to this short exact sequence via the Yoneda correspondence. If $P_\bullet$ is a projective resolution of Q consisting of finitely generated projective modules P_i, then we will call ϵ *$\natural$-phantom* if a cocycle representing ϵ in $\mathrm{Hom}_R(P_1, R)$ is in $\mathrm{Im}(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R))^\natural$ within $\mathrm{Hom}_R(P_1, R)$. We will call α a *$\natural$-phantom extension* of R if ϵ is $\natural$-phantom. We will also call the module M *$\natural$-phantom*.

Since the choice of projective resolution above is not canonical, we must demonstrate that whether $\epsilon \in \mathrm{Ext}^1_R(Q, R)$ is phantom or not is independent of the choice of $P_\bullet$. Let $Q_\bullet$ be another projective resolution of Q consisting of finitely generated projective modules. Given $\epsilon \in \mathrm{Ext}^1_R(Q, R)$ representing the short exact sequence

$$0 \to R \xrightarrow{\alpha} M \to Q \to 0,$$

let $\phi \in \mathrm{Hom}_R(P_1, R)$ (resp., $\phi' \in \mathrm{Hom}_R(Q_1, R)$) be a corresponding cocycle. Assume that $\phi \in \mathrm{Im}(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R))^\natural$ in $\mathrm{Hom}_R(P_1, R)$. We will show that $\phi' \in \mathrm{Im}(\mathrm{Hom}_R(Q_0, R) \to \mathrm{Hom}_R(Q_1, R))^\natural$ in $\mathrm{Hom}_R(Q_1, R)$.

We can lift the identity map $Q \xrightarrow{\mathrm{id}} Q$ to a map of complexes:

(#)

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & Q \arrow[r] & 0 \\ \cdots \arrow[r] & Q_1 \arrow[r] \arrow[u, "f"] & Q_0 \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u, "\mathrm{id}"] & 0 \end{tikzcd}$$

If we let $(-)^\vee$ denote $\mathrm{Hom}_R(-, R)$, then taking the dual of (#) yields the commutative diagram

($\#^\vee$)

$$\begin{tikzcd} \cdots & P_1^\vee \arrow[l] \arrow[d, "f^\vee"] & P_0^\vee \arrow[l] \arrow[d] & Q^\vee \arrow[l] \arrow[d, "\mathrm{id}"] & 0 \arrow[l] \\ \cdots & Q_1^\vee \arrow[l] & Q_0^\vee \arrow[l] & Q^\vee \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Via the Yoneda correspondence, $\phi \mapsto \phi'$ under the map $f^\vee$. Since the element ϕ is in $\mathrm{Im}(P_0^\vee \to P_1^\vee)^\natural_{P_1^\vee}$, applying Axiom (3) to $f^\vee$ gives us

$$\phi' = f^\vee(\phi) \in \mathrm{Im}(P_0^\vee \to Q_1^\vee)^\natural_{Q_1^\vee}.$$

Since $\mathrm{Im}(P_0^\vee \to Q_1^\vee) \subseteq \mathrm{Im}(Q_0^\vee \to Q_1^\vee)$ in $Q_1^\vee$, Axiom (2) shows $\phi' \in \mathrm{Im}(Q_0^\vee \to Q_1^\vee)^\natural_{Q_1^\vee}$, as claimed.

Now that we have a well-defined notion of $\natural$-phantom extensions, we will derive some of the properties of tight closure phantom extensions for $\natural$-phantom extensions using our definition and axioms. We start with a result analogous to [HH5, Proposition 5.7a,e].

Lemma 7.3.3. *Let R be a reduced Noetherian ring, let M be a finitely generated R-module, and let $\alpha : R \to M$ be a $\natural$-phantom extension.*

(a) *If S is any reduced flat R-algebra, then the induced map $S \otimes \alpha : S \to S \otimes_R M$ is a $\natural$-phantom extension of S. In particular, S may be any localization of R.*

(b) *If $\beta : R \to N$ is an injection and $\phi : N \to M$ such that $\alpha = \phi \circ \beta$, then β is also a $\natural$-phantom extension of R.*

Proof. (a) Given the short exact sequence $0 \to R \xrightarrow{\alpha} M \to Q \to 0$, we apply $S \otimes_R -$ to obtain

$$0 \to S \xrightarrow{S\otimes\alpha} S \otimes_R M \to S \otimes_R Q \to 0,$$

which is still exact because S is flat. The element $1 \otimes \epsilon$ in $S \otimes_R \mathrm{Ext}^1_R(Q, R) \cong \mathrm{Ext}^1_S(S \otimes_R Q, S)$ corresponds to this new sequence. Let $P_\bullet$ be a projective resolution of Q as in the definition above. Since S is flat, $S \otimes_R P_\bullet$ remains a projective resolution of Q. As $S \otimes_R \mathrm{Hom}_R(P_i, R) \cong \mathrm{Hom}_S(S \otimes_R P_i, S)$, if ϵ is represented by the cocycle $\phi \in \mathrm{Hom}_R(P_1, R)$, then $1 \otimes \epsilon$ is represented by the image of the cocycle $1 \otimes \phi$ in $\mathrm{Hom}_R(S \otimes_R P_1, S)$. By Axiom (4),

$$1 \otimes \phi \in \mathrm{Im}(\mathrm{Hom}_S(S \otimes_R P_0, S) \to \mathrm{Hom}_S(S \otimes_R P_1, S))^\natural$$

in $\mathrm{Hom}_S(S \otimes_R P_1, S)$. Therefore, $S \otimes \alpha$ is a $\natural$-phantom extension of S.

(b) From the injective maps α and β, where α is injective since it is $\natural$-phantom, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r, "\pi"] & Q \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r, "\beta"] \arrow[u, "\mathrm{id}"'] & N \arrow[r, "\pi'"] \arrow[u, "\phi"'] & Q' \arrow[r] \arrow[u, "\overline{\phi}"'] & 0
\end{tikzcd}$$

where $\overline{\phi}(u+\beta(R)) = \phi(u)+\alpha(R)$. Let $\epsilon \in \mathrm{Ext}^1_R(Q,R)$ correspond to the top row, and let $\epsilon' \in \mathrm{Ext}^1_R(Q',R)$ correspond to the bottom row via the Yoneda correspondence. The map $\overline{\phi}$ induces a map $\psi : \mathrm{Ext}^1_R(Q,R) \to \mathrm{Ext}^1_R(Q',R)$. By [Mac, Lemma III.1.2], $\psi(\epsilon)$ corresponds to the short exact sequence $0 \to R \to N' \to Q' \to 0$, where

$$N' := \{(m,\overline{n}) \in M \oplus Q' \,|\, \pi(m) = \overline{\phi}(\overline{n})\}.$$

We claim that N is isomorphic to N'. Indeed, let $f : N \to N'$ be the map

$$f(n) = (\phi(n), \pi'(n)) = (\phi(n), \overline{n}).$$

The target of f is N' as $\pi(\phi(n)) = \overline{\phi}(\pi'(n))$, and f is clearly R-linear. If $f(n) = (0,0)$, then $\overline{n} = 0$ implies that $n = \beta(r)$, for some $r \in R$. Then $\alpha = \phi \circ \beta$ implies that $\phi(n) = \alpha(r)$, but $\phi(n) = 0$ and α injective means that $r = 0$ so that $n = \beta(0) = 0$. Therefore, f is injective. Given $(m,\overline{n})$ in N', we have $f(n) = (\phi(n), \overline{n})$. Since $\pi(m) = \overline{\phi}(\overline{n})$, $\pi(m - \phi(n)) = 0$, and $m - \phi(n) = \alpha(r)$, for some $r \in R$. Thus,

$$f(n + \beta(r)) = (\phi(n+\beta(r)), \overline{n}) = (\phi(n) + \alpha(r), \overline{n}) = (m, \overline{n}),$$

and so f is also surjective.

Therefore, $\psi(\epsilon) = \epsilon'$. If $P_\bullet$ is a projective resolution of M consisting of finitely generated projective modules, and $P'_\bullet$ is a projective resolution of N consisting of finitely generated projective modules, then by hypothesis, a cocycle representing ϵ in $\mathrm{Hom}_R(P_1,R)$ is also in $\mathrm{Im}(\mathrm{Hom}_R(P_0,R) \to \mathrm{Hom}_R(P_1,R))^\natural$. We can lift $\overline{\phi} : Q' \to Q$ to a map of complexes $P'_\bullet \to P_\bullet$. After applying $\mathrm{Hom}_R(-,R)$, we get the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_R(P_1,R) \arrow[d, "g"'] & \mathrm{Hom}_R(P_0,R) \arrow[l] \arrow[d] \\ \mathrm{Hom}_R(P'_1,R) & \mathrm{Hom}_R(P'_0,R) \arrow[l] \end{tikzcd}$$

where the map g induces $\psi : \mathrm{Ext}^1_R(Q,R) \to \mathrm{Ext}^1_R(Q',R)$. Therefore, if we apply Axioms (2) and (3), the cocycle in $\mathrm{Hom}_R(P'_1,R)$ representing $\epsilon' = \psi(\epsilon)$ is also in $\mathrm{Im}(\mathrm{Hom}_R(P'_0,R) \to \mathrm{Hom}_R(P'_1,R))^\natural$. Thus, ϵ' is also phantom. $\square$

We now demonstrate more explicitly what it means for a module to be $\natural$-phantom. In the local case we will derive extra information. Let R be reduced (resp., let $(R,\mathfrak{p})$ be reduced and local). For a finitely generated R-module M and an injection $R \xrightarrow{\alpha} M$, if we set $Q = M/\alpha(R)$ as above, we have the short exact sequence

$$0 \to R \xrightarrow{\alpha} M \to Q \to 0. \tag{7.3.4}$$

Let $w_1,\ldots,w_{n-1}$ be elements of M such that the images $\overline{w_1},\ldots,\overline{w_{n-1}}$ in Q form a generating set for Q (resp., a minimal generating set), and let $w_n = \alpha(1)$ so that $w_1,\ldots,w_n$ generate M. Let

$$\cdots F_2 \to F_1 \xrightarrow{\beta_1} F_0 \xrightarrow{\beta_0} Q \to 0 \tag{7.3.5}$$

be a free resolution of Q (resp., a minimal free resolution), where $F_0 \cong R^{n-1}$ with basis $\epsilon_1,\ldots,\epsilon_{n-1}$ such that β_0 is given by $\epsilon_i \mapsto \overline{w_i}$. We can also choose a basis for $F_1 \cong R^m$ such that β_1 is given by the $(n-1)\times m$ matrix

$$\beta_1 := \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,m} \end{pmatrix}$$

(where the entries b_{ij} are in $\mathfrak{p}$ when (7.3.5) is a minimal resolution over $(R, \mathfrak{p})$). We can then construct the diagram

(7.3.6)
$$\begin{tikzcd} R^m \arrow[r, "\gamma_1"] \arrow[d, "\mathrm{id}"] & R^n \arrow[r, "\gamma_0"] \arrow[d, "\pi"] & M \arrow[r] \arrow[d] & 0 \\ R^m \arrow[r, "\beta_1"] & R^{n-1} \arrow[r, "\beta_0"] & Q \arrow[r] & 0 \end{tikzcd}$$

where R^n has basis $\epsilon_1, \ldots, \epsilon_n$, $\pi(\epsilon_i) = \epsilon_i$ for $i < n$, and $\pi(\epsilon_n) = 0$. The map γ_0 is given by $\epsilon_i \mapsto w_i$, and γ_1 is given by the $n \times m$ matrix

$$\gamma_1 := \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,m} \\ b_{n1} & \cdots & b_{nm} \end{pmatrix},$$

where $b_{nj}w_n + b_{1j}w_1 + \cdots + b_{n-1,j}w_{n-1} = 0$ in M, for $1 \leq j \leq m$. (Such a b_{nj} exists for all j because $b_{1j}\overline{w_1} + \cdots + b_{n-1,j}\overline{w_{n-1}} = 0$ in $Q = M/Rw_n$.)

From the construction, it is clear that (7.3.6) commutes and that $\gamma_0 \circ \gamma_1 = 0$. The choice of the w_i implies that (7.3.6) is exact at M. To see that $\ker \gamma_0 \subseteq \mathrm{Im}\ \gamma_1$, suppose that $r_1 w_1 + \cdots r_n w_n = 0$ in M. Then $r_1\overline{w_1} + \cdots + r_{n-1}\overline{w_{n-1}} = 0$ in Q, and so there exist $s_1, \ldots, s_m$ in R such that

$$\beta_1(s_1, \ldots, s_m)^{\mathrm{tr}} = (r_1, \ldots, r_{n-1})^{\mathrm{tr}},$$

where $(-)^{\mathrm{tr}}$ denotes the transpose of a matrix. Then

$$\gamma_1(s_1, \ldots, s_m)^{\mathrm{tr}} = (r_1, \ldots, r_{n-1}, r)^{\mathrm{tr}} \in R^n.$$

Since $\gamma_0 \circ \gamma_1 = 0$, we see that $r_1 w_1 + \cdots + r_{n-1} w_{n-1} + r w_n = 0$ in M. Therefore our hypothesis implies that $r w_n = r_n w_n$, and so

$$\alpha(r - r_n) = (r - r_n)\alpha(1) = (r - r_n)w_n = 0.$$

Since α is injective, $r = r_n$ so that the vector $(r_i) \in \operatorname{Im} \gamma_1$.

We can now conclude that the top row of (7.3.6) is a finite free presentation of M. (Since we do not know *a priori* whether $w_1, \ldots, w_n$ form a minimal basis for M in the local case, we do not know whether the b_{nj} are in $\mathfrak{p}$, i.e., we do not know whether our presentation of M is minimal when R is local.)

We also obtain a commutative diagram

(7.3.7)
$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r] & Q \arrow[r] & 0 \\
F_2 \arrow[r] \arrow[u] & R^m \arrow[r, "\beta_1"] \arrow[u, "\phi"'] & R^{n-1} \arrow[r, "\beta_0"] \arrow[u, "\psi"'] & Q \arrow[r] \arrow[u, "\mathrm{id}"'] & 0
\end{tikzcd}$$

where $\psi(\epsilon_i) = w_i$, and ϕ is given by the $1 \times m$ matrix $(-b_{n1} \cdots - b_{nm})$. Because $b_{1j}w_1 + \cdots + b_{nj}w_n = 0$ in M, for $1 \le j \le m$, it is clear that (7.3.7) commutes as claimed. We can then take the dual of (7.3.7) into R:

(7.3.8)
$$\begin{tikzcd}
0 \arrow[d] & R \arrow[l] \arrow[d, "\phi^{\mathrm{tr}}"] & \operatorname{Hom}_R(M,R) \arrow[l] \arrow[d] & \operatorname{Hom}_R(Q,R) \arrow[l] \arrow[d, "\mathrm{id}"] & 0 \arrow[l] \\
F_2 & R^m \arrow[l] & R^{n-1} \arrow[l, "\beta_1^{\mathrm{tr}}"'] & \operatorname{Hom}_R(Q,R) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Let Z be the kernel of $R^m \cong \operatorname{Hom}_R(R^m, R) \to \operatorname{Hom}_R(F_2, R) \cong F_2$, and let B be the image of β_1^{tr}. Then an element of $\operatorname{Ext}^1_R(Q, R)$ is an element of Z/B. In fact, as described in [HH5, Discussion 5.5], the element of $\operatorname{Ext}^1_R(Q, R)$ corresponding to the short exact sequence (7.3.4) is represented by the map $\phi : R^m \to R$ in (7.3.7). Equivalently, it is represented by the image of ϕ^{tr} in (7.3.8).

We are now ready to state an equivalent condition for a finitely generated R-module M to be $\natural$-phantom over a local ring.

Lemma 7.3.9. *Let R be a reduced ring, let M be a finitely generated module, and let $\alpha : R \to M$ be an injective map. Using the notation of the preceding discussion, α is a $\natural$-phantom extension of R if and only if the vector $(b_{n1}, \ldots, b_{nm})^{\mathrm{tr}}$ is in $B^{\natural}_{R^m}$,* *where B is the R-span in R^m of the vectors $(b_{i1}, \dots, b_{im})^{\mathrm{tr}}$, for $1 \le i \le n-1$, and $(-)^{\mathrm{tr}}$ denotes transpose.*

If, moreover, $(R, \mathfrak{p})$ is local, then we can choose the b_{ij} to be in $\mathfrak{p}$, for $1 \le i \le n-1$ and all j.

Proof. By our definition and the constructions above, α is $\natural$-phantom if and only if the cocycle representing the corresponding element ϵ in $\mathrm{Ext}^1_R(Q, R)$ is in $(\mathrm{Im}\ \beta_1^{\mathrm{tr}})^{\natural}_{R^m}$. As pointed out above, ϵ is represented by the image of ϕ^{tr}, which is $(-b_{n1}, \dots, -b_{nm})^{\mathrm{tr}}$. Moreover, the image of β_1^{tr} is the R-span of the row vectors of β_1, which is the R-span of $(b_{i1}, \dots, b_{im})^{\mathrm{tr}}$, for $1 \le i \le n-1$.

The second claim follows from the parenthetical remarks in the discussion just above. $\square$

We can now prove the following fact, an analogue of [HH5, Proposition 5.14].

Lemma 7.3.10. *Let R be reduced, and let M be a finitely generated R-module. If $\alpha : R \to M$ is a $\natural$-phantom extension, then the image of $\alpha(1)$ is not in $\mathfrak{p}M_{\mathfrak{p}}$, for any prime ideal $\mathfrak{p}$ of R.*

Proof. Suppose there exists a prime $\mathfrak{p}$ such that $\alpha(1) \in \mathfrak{p}M$. This remains true when we localize at $\mathfrak{p}$, and by Lemma 7.3.3(a), $\alpha_{\mathfrak{p}}$ is a $\natural$-phantom extension of $R_{\mathfrak{p}}$. Therefore, we can assume that $(R, \mathfrak{p})$ is a reduced local ring. Using the notation above developed for the reduced local case, $\alpha(1) = w_n$. So, $\alpha(1) \in \mathfrak{p}M$ if and only if $w_n = r_1 w_1 + \cdots + r_{n-1} w_{n-1}$ such that the r_i are in $\mathfrak{p}$. This occurs if and only if the vector $(-r_1, \dots, -r_{n-1}, 1)^{\mathrm{tr}}$ is in $\ker(R^n \to M) = \mathrm{Im}(R^m \to R^n) = \mathrm{Im}\ \gamma_1$. In order for a vector with last component a unit to be in $\mathrm{Im}\ \gamma_1$, the last row of γ_1 must generate the unit ideal, i.e., $(b_{n1}, \dots, b_{nm})R = R$. Therefore, there exists j_0 such that $b_{nj_0} \in R \setminus \mathfrak{p}$. By Lemma 7.3.9, since α is $\natural$-phantom, the vector $(b_{n1}, \dots, b_{nm})^{\mathrm{tr}}$ is in

$B^{\natural}_{R^m}$, where B is the R-span in R^m of the vectors $(b_{i1}, \ldots, b_{im})^{\mathrm{tr}}$, for $1 \leq i \leq n-1$. Since R is local, we may assume that every b_{ij}, for $1 \leq i \leq n-1$ and $1 \leq j \leq m$, is in $\mathfrak{p}R$, Axiom (2) implies that $(b_{n1}, \ldots, b_{nm})^{\mathrm{tr}}$ is in $(\mathfrak{p}R^m)^{\natural}_{R^m}$. Using Axiom (3), for the projection $R^m \to R$ mapping onto the j_0^{th}-coordinate, we see that $b_{nj_0} \in (\mathfrak{p}R)^{\natural}_R$, but $(\mathfrak{p}R)^{\natural}_R = \mathfrak{p}$ by Lemma 7.2.2. This implies that b_{nj_0} cannot be a unit, and so $\alpha(1) \notin \mathfrak{p}M$. $\square$

In order to use $\natural$-phantom extensions to produce big Cohen-Macaulay modules (resp., algebras) using the previous lemma, we would still need to show that module (resp., algebra) modifications of $\natural$-phantom extensions are still $\natural$-phantom extensions. If this were true, then any finite sequence of modifications of a $\natural$-phantom extension would terminate in a $\natural$-phantom extension. No such sequence could then be a bad sequence because the image of 1 would not be in the expansion of the maximal ideal of R. Therefore, $\natural$-phantom extensions merit further study.

7.4 A Closure Operation Derived From Big Cohen-Macaulay Algebras

Let $\mathscr{R}$ be a suitable category of rings (as defined in Section 7.1), and let $\mathscr{C}$ be the full subcategory of $\mathscr{R}$ of all complete local domains. In this section, we will also assume that for each $(R, m) \in \mathscr{C}$, there exists a directed family $\mathscr{B}(R)$ of big Cohen-Macaulay R-algebras such that

(i) if $B \in \mathscr{B}(R)$ and S is an R-algebra in $\mathscr{R}$, then there exists $C \in \mathscr{B}(S)$ such that the diagram below commutes.

(7.4.1)
$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

(ii) if $S \in \mathscr{C}$ is an R-algebra with $\dim S = \dim R + \dim S/mS$, then $\mathscr{B}(S) \subseteq \mathscr{B}(R)$, where $\mathscr{B}(R)$ directed means that for any $B, C \in \mathscr{B}(R)$, there exists $D \in \mathscr{B}(R)$ such that both B and C map to D.

The hypothesis that $\dim S = \dim R + \dim S/mS$ is equivalent to the condition that the image of every partial system of parameters in R remains part of a system of parameters in S since R and S are complete local. (See [Ho3, Remark 2.7].)

Remark 7.4.2. Theorems 6.6.4 and 6.6.10 of the last chapter show that if we let $\mathscr{R}$ be the category of all rings of positive characteristic and let $\mathscr{B}(R)$ be the class of all big Cohen-Macaulay R-algebras, for each complete local domain R, then the definitions $\mathscr{R}$ and $\mathscr{B}(R)$ satisfy the conditions given above.

We will now define a closure operation, using the big Cohen-Macaulay algebras $\mathscr{B}(R)$, by initially defining it for $\mathscr{C}$ and then extending the definition to all of $\mathscr{R}$. We will then show that this closure operation satisfies the axioms from Section 7.1.

Definition 7.4.3. Let $R \in \mathscr{C}$, and let $N \subseteq M$ be finitely generated R-modules. Then $N_M^\natural$ is the set of all elements $u \in M$ such that $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$, for some $B \in \mathscr{B}(R)$.

Let $S \in \mathscr{R}$, and let $N \subseteq M$ be finitely generated S-modules. Then $N_M^\natural$ is the set of all $u \in M$ such that for all S-algebras T, where $T \in \mathscr{C}$,

$$1 \otimes u \in \operatorname{Im}(T \otimes_S N \to T \otimes_S M)^\natural_{T \otimes_S M}.$$

We will call $N_M^\natural$ the $\natural$-*closure* of N in M.

Before proving that the axioms hold, we must show that the definition of $\natural$-closure for $\mathscr{R}$ is consistent with the definition for $\mathscr{C}$.

Lemma 7.4.4. *If $R \in \mathscr{C}$, and $N \subseteq M$ are finitely generated R-modules, then $N_M^\natural$ is independent of which definition above is applied.*

Proof. Let $u \in M$. If $1 \otimes u \in \operatorname{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T\otimes_R M}$ for all R-algebras T, such that $T \in \mathscr{C}$, then this is certainly true for $T = R$. Therefore, membership defined by the second criterion implies the first.

Now suppose that there exists $B \in \mathscr{B}(R)$ such that $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$. Let S be an R-algebra such that $S \in \mathscr{C}$. Then there exists $C \in \mathscr{B}(S)$ that fills in the commutative square (7.4.1). Hence, the image of u is in

$$\begin{aligned} & \operatorname{Im}(C \otimes_B (B \otimes_R N) \to C \otimes_B (B \otimes_R M)) \\ \cong\ & \operatorname{Im}(C \otimes_R N \to C \otimes_R M) \\ \cong\ & \operatorname{Im}(C \otimes_S (S \otimes_R N) \to C \otimes_S (S \otimes_R M)), \end{aligned}$$

and so $1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}$. Thus, the first criterion also implies the second. $\square$

(7.4.5) For the remainder of this section, R and S are rings in $\mathscr{R}$ with S an R-algebra; I is an ideal of R; and $N \subseteq M$ and W are finitely generated R-modules.

Proposition 7.4.6 (Axiom 1)**.** *With notation as in (7.4.5), the set $N^\natural_M$ is a submodule of M containing N.*

Proof. From the definition, it is clear that it suffices to show the case where $R \in \mathscr{C}$. Let $u, v \in N^\natural_M$, and let $r \in R$. Since $\mathscr{B}(R)$ is directed, there exists a common $B \in \mathscr{B}(R)$ for u and v such that $1 \otimes u$ and $1 \otimes v$ are in $\operatorname{Im}(B \otimes_R N \to B \otimes_R M)$. Therefore, $1 \otimes (ru+v) \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$ as well. This shows that $ru+v \in N^\natural_M$. Finally, if $u \in N$, then $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$, for any $B \in \mathscr{B}(R)$, so that $N^\natural_M$ contains N. $\square$

Proposition 7.4.7 (Axiom 2)**.** *With notation as in (7.4.5), if $M \subseteq W$, then $N^\natural_W \subseteq M^\natural_W$.*

Proof. Let $u \in N^\natural_W$, and let $T \in \mathscr{R}$ be an R-algebra. Then

$$1 \otimes u \in \mathrm{Im}(T \otimes_R N \to T \otimes_R W)^\natural_{T\otimes_R W}.$$

Since $\mathrm{Im}(T \otimes_R N \to T \otimes_R W) \subseteq \mathrm{Im}(T \otimes_R M \to T \otimes_R W)$, it is enough to show the case where $R \in \mathscr{C}$.

Let $R \in \mathscr{C}$, and let $u \in N^\natural_W$. Then there exists $B \in \mathscr{B}(R)$ such that

$$1 \otimes u \in \mathrm{Im}(B \otimes_R N \to B \otimes_R W) \subseteq \mathrm{Im}(B \otimes_R M \to B \otimes_R W).$$

Therefore, $u \in M^\natural_W$. $\square$

Proposition 7.4.8 (Axiom 3). *With notation as in (7.4.5), if $f : M \to W$, then $f(N^\natural_M) \subseteq f(N)^\natural_W$.*

Proof. Let $R \in \mathscr{C}$ initially. Then $u \in N^\natural_M$ implies that there exists $B \in \mathscr{B}(R)$ such that $1 \otimes u \in \mathrm{Im}(B \otimes_R N \to B \otimes_R M)$. This implies that $1 \otimes u = \sum_i b_i \otimes v_i$ such that $b_i \in B$ and $v_i \in N$. Hence $1 \otimes f(u) = \sum_i b_i \otimes f(v_i)$, where $f(v_i) \in f(N)$, for all i. This means that $1 \otimes f(u) \in \mathrm{Im}(B \otimes_R f(N) \to B \otimes_R W)$ so that $f(u) \in f(N)^\natural_W$.

If $S \in \mathscr{R}$ and $u \in N^\natural_M$, then for any S-algebra T, where $T \in \mathscr{C}$,

$$1 \otimes u \in \mathrm{Im}(T \otimes_S N \to T \otimes_S M)^\natural_{T\otimes_S M}.$$

The map $f : M \to W$ induces $T \otimes f : T \otimes_S M \to T \otimes_S W$. Since $T \in \mathscr{C}$, the last paragraph implies that $1 \otimes f(u) \in (T \otimes f)(\mathrm{Im}(T \otimes_S N \to T \otimes_S M))^\natural_{T\otimes_S W}$, but this module is $\mathrm{Im}(T \otimes_S f(N) \to T \otimes_S W)^\natural_{T\otimes_S W}$. Therefore, $f(u) \in f(N)^\natural_W$. $\square$

Proposition 7.4.9 (Axiom 4: persistence). *With notation as in (7.4.5),*

$$\mathrm{Im}(S \otimes_R N^\natural_M \to S \otimes_R M) \subseteq \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}.$$

In particular, if $N = I$ and $M = R$, we have $I^\natural_R S \subseteq (IS)^\natural_S$.

Proof. The second claim follows directly from the first claim. For the first, let $R \to S$ be a map such that R and S are in $\mathscr{R}$, and let $N \subseteq M$ be finitely generated R-modules with $u \in N^\natural_M$. We must show that $1 \otimes u$ in $S \otimes_R M$ is an element of $\operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}$.

Suppose that $T \in \mathscr{C}$ is an S-algebra. Then T is also an R-algebra. Since $u \in N^\natural_M$, we have $1 \otimes u \in \operatorname{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T\otimes_R M}$. Since the functors $T \otimes_R -$ and $T \otimes_S (S \otimes_R -)$ are isomorphic, we also have that

$$1 \otimes (1 \otimes u) \in \operatorname{Im}(T \otimes_S (S \otimes_R N) \to T \otimes_S (S \otimes_R M))^\natural$$

in $T \otimes_S (S \otimes_R M)$, so that $1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}$ as required. □

Proposition 7.4.10 (Axiom 5)**.** *With notation as in (7.4.5),* $I^\natural_R N^\natural_M \subseteq (IN)^\natural_M$.

Proof. By a straightforward application of Axioms (9), (10), and (8), we may assume that R is a complete local domain in $\mathscr{C}$. If $u \in I^\natural_R N^\natural_M$, then $u = \sum_i a_i v_i$ such that $a_i \in I^\natural_R$ and $v_i \in N^\natural_M$. Since $\mathscr{B}(R)$ is directed, there exists a single $B \in \mathscr{B}(R)$ such that $a_i \in IB$ and $1 \otimes v_i \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$ for all i. Thus, $a_i = \sum_k r_{ik} c_{ik}$, where $r_{ik} \in I$ and $c_{ik} \in B$, for all i and k, and $1 \otimes v_i = \sum_j b_{ij} \otimes w_{ij}$, where $b_{ij} \in B$ and $w_{ij} \in N$, for all i and j.

In $B \otimes_R M$, we have

$$\begin{aligned} 1 \otimes u &= 1 \otimes \sum_i a_i v_i = \sum_i a_i (1 \otimes v_i) \\ &= \sum_i (\sum_k r_{ik} c_{ik})(\sum_j b_{ij} \otimes w_{ij}) = \sum_{i,j,k} (c_{ik} b_{ij}) \otimes (r_{ik} w_{ij}), \end{aligned}$$

where $c_{ik} b_{ij} \in B$ and $r_{ik} w_{ij} \in IN$. Thus, $1 \otimes u \in \operatorname{Im}(B \otimes_R IN \to B \otimes_R M)$, so $u \in (IN)^\natural_M$. □

Proposition 7.4.11 (Axiom 6)**.** *With notation as in (7.4.5),* $(N^\natural_M)^\natural_M = N^\natural_M$.

Proof. Let $T \in \mathscr{C}$ be an R-algebra. If $u \in (N^\natural_M)^\natural_M$, then

$$1 \otimes u \in \operatorname{Im}(T \otimes_R N^\natural_M \to T \otimes_R M)^\natural_{T\otimes_R M}.$$

By Axioms (3) and (4),

$$1 \otimes u \in (\operatorname{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T\otimes_R M})^\natural_{T\otimes_R M}.$$

Thus, if the claim is true for $\mathscr{C}$, then $1 \otimes u \in \operatorname{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T\otimes_R M}$, and so $u \in N^\natural_M$.

Hence, we assume that $R \in \mathscr{C}$, and $u \in (N^\natural_M)^\natural_M$. Then

$$1 \otimes u \in \operatorname{Im}(B \otimes_R N^\natural_M \to B \otimes_R M),$$

for some $B \in \mathscr{B}(R)$. This means that $1 \otimes u = \sum_i b_i \otimes v_i$, where $b_i \in B$ and $v_i \in N^\natural_M$ for all i. For each i, there exists $C_i \in \mathscr{B}(R)$ such that $1 \otimes v_i \in \operatorname{Im}(C_i \otimes_R N \to C_i \otimes_R M)$. Since $\mathscr{B}(R)$ is directed, there exists $C \in \mathscr{B}(R)$ that is a B-algebra and a C_i-algebra for all i, with $f : B \to C$. We can then write $1 \otimes v_i = \sum_j c_{ij} \otimes w_{ij}$, where $c_{ij} \in C$ and $w_{ij} \in N$, for all i and j. As an element of $C \otimes M$,

$$1 \otimes u = \sum_i f(b_i) \otimes v_i = \sum_i \sum_j f(b_i)c_{ij} \otimes w_{ij},$$

which is an element of $\operatorname{Im}(C \otimes_R N \to C \otimes_R M)$. Thus, $u \in N^\natural_M$. $\square$

Proposition 7.4.12 (Axiom 7)**.** *With notation as in (7.4.5), if* $N^\natural_M = N$, *then* $0^\natural_{M/N} = 0$.

Proof. For $R \in \mathscr{C}$, if $\overline{u} := u + N \in 0^\natural_{M/N}$, then there exists $B \in \mathscr{B}(R)$ such that $1 \otimes \overline{u} = 0$ in $B \otimes_R M/N$. Since $B \otimes_R M/N \cong B \otimes_R M/\operatorname{Im}(B \otimes_R N \to B \otimes_R M)$,

$$1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M).$$

Thus, $u \in N^\natural_M = N$, and $\overline{u} = 0$.

For $R \in \mathscr{R}$, if $\overline{u} \in 0^\natural_{M/N}$, then for every R-algebra T in $\mathscr{C}$, $1 \otimes \overline{u} \in 0^\natural$ in $T \otimes_R M/N$, calculated over T. Using $T \otimes_R M/N \cong T \otimes_R M/\mathrm{Im}(T \otimes_R N \to T \otimes_R M)$, the image of $1 \otimes u$ in $T \otimes_R M/\mathrm{Im}(T \otimes_R N \to T \otimes_R M)$ is in $0^\natural$. By Axiom (3), the image of $1 \otimes u$ in $T \otimes_R M/\mathrm{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T \otimes_R M}$ is also in $0^\natural$. By Axiom (6) and the previous paragraph, $0^\natural = 0$ here, so $1 \otimes u \in \mathrm{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T \otimes_R M}$ for all such T. Hence, $u \in N^\natural_M = N$, and $\overline{u} = 0$. □

Proposition 7.4.13 (Axiom 8)**.** *With notation as in (7.4.5), if $u + \mathfrak{p}M$ is in $\mathrm{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)^\natural_{M/\mathfrak{p}M}$, calculated over $R/\mathfrak{p}R$, for all minimal primes $\mathfrak{p}$ of R, then $u \in N^\natural_M$.*

Proof. Let $R \in \mathscr{R}$, and let T be an R-algebra in $\mathscr{C}$. The kernel of $R \to T$ is a prime ideal, so there exists a minimal prime $\mathfrak{p}$ of R such that the map $R \to T$ factors through $R/\mathfrak{p}$. Then $\overline{u} \in \mathrm{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)^\natural_{M/\mathfrak{p}M}$ implies that

$$1 \otimes \overline{u} \in \mathrm{Im}(T \otimes_{R/\mathfrak{p}} N/\mathfrak{p}N \to T \otimes_{R/\mathfrak{p}} M/\mathfrak{p}M)^\natural,$$

in $T \otimes_{R/\mathfrak{p}} M/\mathfrak{p}M$, since T is an $R/\mathfrak{p}$-algebra. This implies that

$$1 \otimes u \in \mathrm{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T \otimes_R M}.$$

Therefore, $u \in N^\natural_M$. □

Proposition 7.4.14 (Axiom 9)**.** *With notation as in (7.4.5), if $u/1 \in (N_m)^\natural_{M_m}$, calculated over R_m, for all maximal ideals m of R, then $u \in N^\natural_M$.*

Proof. Let $R \in \mathscr{R}$, and let (T, n) be an R-algebra in $\mathscr{C}$. If m is a maximal ideal of R, then $f : R \to T$ factors through R_m if and only if $f^{-1}(n) \subseteq m$. If $R \to T$ does factor through any such R_m, then T is also an R_m-algebra, and $u/1 \in (N_m)^\natural_{M_m}$ implies that $1 \otimes u/1 \in \mathrm{Im}(T \otimes_{R_m} N_m \to T \otimes_{R_m} M_m)^\natural$ in $T \otimes_{R_m} M_m$. This in turn implies that $1 \otimes u \in \mathrm{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T \otimes_R M}$.

Thus, $u \in N_M^\natural$ if, for every such R-algebra T, there exists a maximal ideal m such that $f^{-1}(n) \subseteq m$. Suppose to the contrary that there is a T such that $f^{-1}(n) \nsubseteq m$, for all maximal ideals m of R. Then $f^{-1}(n)$ must be the unit ideal, which implies that $T = nT$, a contradiction since T is a local Noetherian ring. $\square$

Proposition 7.4.15 (Axiom 10)**.** *With notation as in (7.4.5), let S be faithfully flat over R. If*

$$1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S \otimes_R M},$$

calculated over S, then $u \in N_M^\natural$. In particular, if $N = I$ and $M = R$, we have $(IS)_S^\natural \cap R \subseteq I_R^\natural$.

Proof. The second claim follows directly from the first. Let $u \in M$ be such that $1 \otimes u$ is in $\operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S \otimes_R M}$. By Axiom (9), it is enough to show $u \in (N_m)^\natural_{M_m}$, for all maximal ideals m of R. For any such m, since S is faithfully flat over R, there exists a prime n of S lying over m. Then $R_m \to S_n$ is a faithfully flat local map. Using Axiom (4), we can then assume, without loss of generality, that R and S are local and $R \to S$ is a local map.

Suppose that the claim is true when both rings are complete local. Using persistence again, for the map $S \to \widehat{S}$, and using the fact that $\widehat{R} \to \widehat{S}$ is still a faithfully flat map, we can conclude that $u \in \widehat{N}^\natural_{\widehat{M}}$, calculated over $\widehat{R}$. If T is an R-algebra in $\mathscr{C}$, then T is also an $\widehat{R}$-algebra. Therefore,

$$1 \otimes u \in \operatorname{Im}(T \otimes_{\widehat{R}} (\widehat{R} \otimes_R N) \to T \otimes_{\widehat{R}} (\widehat{R} \otimes_R M))^\natural$$

in $T \otimes_{\widehat{R}} (\widehat{R} \otimes_R M)$, and so $1 \otimes u \in \operatorname{Im}(T \otimes_R N \to T \otimes_R M)^\natural_{T \otimes_R M}$ as needed.

We can now assume that R and S are complete local rings and that $R \to S$ is a faithfully flat local map. It suffices to show that $\overline{u} \in \operatorname{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)^\natural_{M/\mathfrak{p}M}$, for all minimal primes $\mathfrak{p}$ of R, by Axiom (8). Let $\mathfrak{p}$ be a minimal prime of R. Then $S/\mathfrak{p}S$

is still faithfully flat over $R/\mathfrak{p}$, and we can choose $\mathfrak{q}$ a prime of S that is minimal over $\mathfrak{p}S$ such that $\dim S/\mathfrak{q} = \dim S/\mathfrak{p}S$. Then

$$\dim S/\mathfrak{q} = \dim S/\mathfrak{p}S = \dim R/\mathfrak{p} + \dim(S/\mathfrak{p}S)/m(S/\mathfrak{p}S)$$

by the faithful flatness of $S/\mathfrak{p}S$ over $R/\mathfrak{p}$. We can then conclude that

$$\dim S/\mathfrak{q} = \dim R/\mathfrak{p} + \dim(S/\mathfrak{q})/m(S/\mathfrak{q}).$$

Using persistence for the map $S \to S/\mathfrak{q}$ with Axiom (8), we can assume that R and S are complete local domains in $\mathscr{C}$ such that $\dim S = \dim R + \dim S/mS$.

Now, $1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M}$ implies that there exists a big Cohen-Macaulay S-algebra B in $\mathscr{B}(S)$ such that the image of $1 \otimes u$ is in

$$\operatorname{Im}(B \otimes_S (S \otimes_R N) \to B \otimes_S (S \otimes_R M)).$$

Therefore, $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$, where $B \in \mathscr{B}(R)$ by condition (ii) from the start of the section, and so $u \in N^\natural_M$. $\square$

Remark 7.4.16. The method used in the preceding proof to reduce from the complete local case to the complete local domain case is essentially the same argument used by Hochster in [Ho3, Lemma 3.6c].

Proposition 7.4.17 (Axiom 11)**.** *With notation as in (7.4.5), let S be a module-finite extension of R. If*

$$1 \otimes u \in \operatorname{Im}(S \otimes_R N \to S \otimes_R M)^\natural_{S\otimes_R M},$$

calculated over S, then $u \in N^\natural_M$.

Proof. By Axiom (9), it is enough to check the claim at each maximal ideal m of R. Pairing this with Axiom (4) for the map $S \to (R \setminus m)^{-1}S$, we may assume without loss of generality that (R, m) is local, every maximal ideal of S lies over m, and $R \to S$ is still a module-finite extension.

If we then use persistence for the map $S \to \widehat{S}$ (where $\widehat{S}$ is the m-adic completion of S), Axiom (10) for the faithfully flat map $R \to \widehat{R}$, and the fact that $\widehat{S} \cong \widehat{R} \otimes_R S$, we can assume that R is complete local. If we now use Axiom (4) for $S \to S/\mathfrak{q}$ and Axiom (8), where $\mathfrak{q}$ is a minimal prime of S lying over a minimal prime $\mathfrak{p}$ of R, then we may assume that (R, m) is a complete local domain, S is a module-finite extension domain, $\dim R = \dim S$, and every maximal ideal of S lies over m.

Since there exists an S-algebra T such that T is a complete local domain with $\dim T = \dim S$ obtained by localizing S at a maximal ideal, completing, and then killing a minimal prime, T is a local R-algebra with $\dim T = \dim R + \dim T/mT$. By condition (ii) from the beginning of the section, $\mathscr{B}(T) \subseteq \mathscr{B}(R)$.

Let $u \in M$ such that $1 \otimes u \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^{\natural}_{S \otimes_R M}$. Then

$$1 \otimes u \in \mathrm{Im}(T \otimes_R N \to T \otimes_R M)^{\natural}_{T \otimes_R M},$$

using the isomorphism of $T \otimes_S (S \otimes_R -)$ and $T \otimes_R -$, and so there exists $B \in \mathscr{B}(T)$ such that the image of $1 \otimes u$ is in $\mathrm{Im}(B \otimes_T (T \otimes_R N) \to B \otimes_T (T \otimes_R M))$, which is isomorphic to $\mathrm{Im}(B \otimes_R N \to B \otimes_R M)$. Hence, $u \in N^{\natural}_M$ as B is also in $\mathscr{B}(R)$. $\square$

Proposition 7.4.18 (Axiom 12)**.** *With notation as in (7.4.5), if R is regular, then $N^{\natural}_M = N$.*

Proof. Suppose the claim is true when R is regular local. Then $u \in N^{\natural}_M$ implies, by persistence, that $u/1 \in (N_m)^{\natural}_{M_m}$, for all maximal ideals m of R. Since R_m is regular local, $u/1 \in N_m$, for all m, and this implies that $u \in N$.

We can now assume that (R, m) is regular local. Suppose the claim is true for complete regular local rings. Using Axiom (4), the image of u is in $\widehat{N}^{\natural}_{\widehat{M}} = \widehat{N}$ because

$\widehat{R}$ is regular. Therefore, the image of u in M/N maps to zero via the map

$$\phi : M/N \to \widehat{R} \otimes_R (M/N).$$

Since $\widehat{R}$ is faithfully flat over R, ϕ is injective (see [Mat, 4.c(i)]) so that $u \in N$.

Now we can assume that R is a complete regular local ring and, hence, $R \in \mathscr{C}$. Then $u \in N^\natural_M$ means that there is a $B \in \mathscr{B}(R)$ such that

$$1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M).$$

By Lemma 2.1 of [HH7], B is faithfully flat over R. Therefore, $\overline{u}$ maps to zero under the injective map $M/N \to B \otimes_R (M/N)$, and $u \in N$ by [Mat, 4.c(i)]. □

For the following result and proof, we refer the reader to Theorem 2.2.11 for a statement of phantom acyclicity for tight closure and to [HH1, Section 9] for a general overview of phantom acyclicity. For the topic of K-depth and the generalization of the Buchsbaum-Eisenbud acyclicity criterion used below, see [Ab, Sections 1.1, 1.2] and [Nor, Appendix B].

Proposition 7.4.19 (Axiom 13: phantom acyclicity)**.** *With notation as in (7.4.5), let $R \in \mathscr{C}$, be a complete local domain, and let $G_\bullet$ be a finite free complex over R:*

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0.$$

Let α_i be the matrix map $G_i \to G_{i-1}$, and let r_i be the determinantal rank of α_i, for $1 \le i \le n$. Let b_i be the free rank of G_i, for $0 \le i \le n$. Denote the ideal generated by the size r minors of a matrix map α by $I_r(\alpha)$. If $b_i = r_i + r_{i-1}$ and $\operatorname{ht} I_{r_i}(\alpha_i) \ge i$, for all $1 \le i \le n$, then $Z_i \subseteq (B_i)^\natural_{G_i}$, where B_i is the image of α_{i+1} and Z_i is the kernel of α_i. In other words, if the previous rank and height conditions are satisfied, then the cycles are contained in the $\natural$-closure of the boundaries.

Proof. Choose $B \in \mathscr{B}(R)$. Then $B \otimes_R G_\bullet$ has the same rank condition as $G_\bullet$, and the K-depth of $I_{r_i}(\alpha_i)$ on B is at least ht $I_{r_i}(\alpha_i) \geq i$ since B is a big Cohen-Macaulay algebra over R. By the generalized Buchsbaum-Eisenbud acyclicity criterion (see [Ab, Theorem 1.2.3]), $B \otimes_R G_\bullet$ is acyclic. Therefore,

$$\mathrm{Im}(B \otimes_R Z_i \to B \otimes_R G_i) \subseteq \mathrm{Im}(B \otimes \alpha_{i+1}) = \mathrm{Im}(B \otimes_R B_i \to B \otimes_R G_i),$$

and so if $u \in Z_i$, then $1 \otimes u \in \mathrm{Im}(B \otimes_R B_i \to B \otimes_R G_i)$, which says that $u \in (B_i)^\natural_{G_i}$. $\square$

We have now verified that all axioms of Section 7.1 are satisfied by the closure operation we defined based on the existence of big Cohen-Macaulay R-algebras for complete local domains.

CHAPTER 8

Open Questions

In this final chapter, we present several questions that are left for future study.

Question 8.1. Let R be a standard graded K-algebra domain of positive characteristic p such that K is algebraically closed, and let m be the homogeneous maximal ideal. Can the equivalence of tight closure and graded-plus closure for finitely generated modules $N \subseteq M$ such that M/N is m-coprimary be used to prove the equivalence of the closure operations for *all* finitely generated modules?

Can this be done if one additionally assumes that the ideal of all test elements is m or is m-primary, as is the case for the cubical cone?

Question 8.2. What is the injective hull $E_{A^\infty}(K^\infty)$, when $A = K[x_1, \ldots, x_n]$ or $A = K[[x_1, \ldots, x_n]]$, $n \geq 2$, and K is a field of positive characteristic? Is $E_{A^\infty}(K^\infty)$ the module of all formal sums with DCC support as described in Section 3.3?

Question 8.3. Is every solid algebra S over a complete local domain R of positive characteristic a seed over R?

Question 8.4. Are the main results of Chapter 6 true in the equal characteristic 0 case? In particular, if R contains a copy of $\mathbb{Q}$, are integral extensions of seeds over R still seeds over R? If R is also a complete local domain, are tensor products of seeds over R still seeds? If S is also a complete local domain and an R-algebra, does any seed over R map to a seed over S?

Can these equal characteristic 0 results be achieved using a reduction to characteristic p argument?

Question 8.5. Theorem 6.4.8 shows that if R is a local Noetherian ring of positive characteristic and S is a seed domain over R, then the absolute integral closure S^+ of S is still a seed over R. Is S^+ actually a big Cohen-Macaulay algebra over R, at least when R is an excellent local domain?

A positive answer to this last question would greatly generalize Hochster and Huneke's theorem in [HH2] that R^+ is a big Cohen-Macaulay algebra over R when R is an excellent local domain.

Question 8.6. When R is a complete regular local ring of positive characteristic, what are the minimal seeds over R?

In [Ho3], Hochster asks the same question for minimal solid algebras. If one could show that these minimal seeds are in fact integral extensions of R, then an immediate consequence would be the equivalence of tight closure and plus closure.

Question 8.7. Let (R, m) be an excellent local domain of positive characteristic. If B is a big Cohen-Macaulay R-algebra and also an R^+-algebra, is B faithfully flat over R^+? What if B is m-adically separated or absolutely integrally closed?

A positive answer to either of the previous questions would show that tight closure equals plus closure. The results of Chapter 6 allow us to only check the case where B is m-adically separated and absolutely integrally closed.

Question 8.8. Are the axioms of Chapter 7 strong enough to imply some form of the Briançon-Skoda Theorem, at least for regular rings?

Question 8.9. Are the axioms of Chapter 7 strong enough to imply the weakly functorial existence of big Cohen-Macaulay algebras over complete local domains? Are they strong enough to imply any existence result for big Cohen-Macaulay algebras or modules?

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