Research Statement

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1. Introduction

My research interests reside in commutative algebra, specifically tight closure theory and the existence of big Cohen-Macaulay modules and algebras. Much of my research has focused on rings containing a field of positive prime characteristic.

The theory of commutative rings, developed over the last century, serves as a fundamental tool for modern mathematicians studying algebraic geometry and algebraic number theory. Commutative rings, especially those containing a field, provide the "local data" used by algebraic geometers to construct and study algebraic varieties (algebraic analogues of complex manifolds from differential geometry). As a result, commutative algebra lies at the very heart of modern algebraic geometry. For many algebraic number theorists, commutative algebra is also an indispensable tool, as many problems revolve around the study of commutative rings containing the integers. In the last several decades, commutative algebra, via the study of Cohen-Macaulay rings and homological algebra, has also provided new and significant results in algebraic combinatorics.

Beyond applications to other branches of mathematics, commutative algebra also stands on its own as a very deep and beautiful field of study that possesses many interesting unsolved problems. A number of significant open questions in commutative algebra have been collectively labeled as the "local homological conjectures." Many of these conjectures are actually theorems for rings containing a field and have been proved by numerous people over the last couple of decades. Most of the conjectures are interconnected, and the truth of many of them follows from a *weakly functorial* existence of *big Cohen-Macaulay algebras*.

For a local Noetherian ring R, an R-module M is called a *(balanced) big Cohen-Macaulay (C-M) module* if every system of parameters (s.o.p.) for R is a regular sequence on M. If $M = S$ is an R-algebra, then we call S a *(balanced) big C-M algebra* for R. If R has Krull dimension d, a set of elements $x_1, \dots, x_d$ is called a *system of parameters* if the dimension of $R/(x_1, \dots, x_d)$ is zero. Given an R-module M (not necessarily finitely generated) and elements $y_1, \dots, y_n$ in R, one calls $y_1, \dots, y_n$ a *regular sequence* on M if y_1 is not a zerodivisor on M, y_{i+1} is not a zerodivisor on $M/(y_1, \dots, y_i)M$, for all $i \geq 1$, and $M/(y_1, \dots, y_n)M \neq 0$.

Following the terminology of M. Hochster and C. Huneke in [HH4], a category of rings enjoys a *weakly functorial* existence of big C-M algebras, if a given map of rings $R \to S$ in the category can be extended to a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B is a big C-M algebra for R and C is a big C-M algebra for S. In [HH4], Hochster and Huneke showed that such a weakly functorial existence of big C-M algebras occurs for rings containing a field. The core of their proof is their remarkable theorem (see [HH2]) that R^+, the integral closure of R in an algebraic closure of its fraction field, is a big C-M algebra for R when R is an excellent local domain of positive prime characteristic. If R is $\mathbb{N}$-graded, they also construct a $\mathbb{Q}$-graded domain $R^{+\mathrm{GR}}$ inside R^+ and show that every homogeneous system of parameters is a regular sequence on $R^{+\mathrm{GR}}$.

A closely related topic is the theory of tight closure developed by Hochster and Huneke. Given a domain R containing a field of positive characteristic p and an ideal I, the *tight closure* of I in R is

$$I^*_R := \{u \in R \,|\, \exists c \neq 0 \text{ s.t. } cu^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\},$$

where $I^{[p^e]}$ is the ideal

$$(a^{p^e} \,|\, a \in I)R.$$

For a detailed introduction to tight closure, including the definition of the tight closure N^*_M for finitely generated R-modules $N \subseteq M$ and for rings that are not domains, see [HH].

Like the existence of big C-M algebras, tight closure theory can give (often surprisingly simple) proofs of results that lead to positive solutions of many of the homological conjectures when the ring contains a field. There are still many open questions surrounding tight closure theory and big C-M modules and algebras, even in positive prime characteristic. My research has concentrated on studying some of the open problems related to these two areas.

2. Seeds and Big Cohen-Macaulay Algebras

I have been studying algebras that map to a big C-M R-algebra over a local ring (R, m). I call these *seed algebras* over R. See [D1] or [D3].

One of the most important results about seeds that I have proved is that an integral extension of a seed is still a seed over a local ring of positive characteristic. This result can be used to imply the existence of big C-M algebras over complete local domains of positive characteristic and can be viewed as a generalization of previously known existence results for big C-M algebras. I have also been able to show that all seeds can be mapped to an absolutely integrally closed, m-adically separated, quasilocal big C-M algebra domain.

In developing the properties of seeds I have also been able to prove some new results about big C-M algebras over a complete local domain R of positive characteristic. Specifically, if B and B' are big C-M R-algebras, then B and B' map to a common big C-M R-algebra C,

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & B' \end{array}$$

showing that the class of big C-M algebras over R forms a directed system. I have also proved that if $R \to S$ is a local map of positive characteristic complete local domains, and B is a big C-M algebra over R, then there exists a big C-M algebra C over S giving a commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

This result generalizes the weakly-functorial existence result of Hochster and Huneke for big C-M algebras. The two results above can then be used to define a closure operation using big C-M algebras over R. Hochster's result in [Ho] shows that this operation is equivalent to tight closure for complete local domains, but my results show *directly* that the closure operation derived from big C-M algebras has many of the nice properties of tight closure.

Question 2.1. Are the seed and big C-M algebra results above also true in equal characteristic zero?

Two ways to approach this question would be to use either the descent methods employed by Hochster and Huneke in their development of characteristic zero tight closure (see [HH6]), or to use the *Lefschetz hull* and ultraproduct methods of Aschenbrenner and Schoutens (see [AS]) to transfer positive characteristic results to equal characteristic zero.

Question 2.2. Can the notion of a seed being extended to R-modules?

Early work has shown we can generalize seeds to include modules that can be mapped to big C-M modules that possess a Frobenius action. Since big C-M algebras in positive characteristic have such an action via the Frobenius endomorphism, this notion does generalize our earlier notion of seeds. Although not every big C-M module in positive characteristic has a Frobenius action, it is interesting to ask the following question as a positive answer would provide for a very satisfactory notion of seeds for R-modules.

Question 2.3. Can every big C-M module in positive characteristic be mapped to a big C-M module that has a Frobenius action?

3. Axioms for a "Good" Closure Operation

A recent problem I have studied revolves around finding a set of axioms that will yield a closure theory for ideals and modules that is strong enough to imply the existence of big C-M modules or algebras for complete local domains without constraint on characteristic.

For the case of big C-M modules, I have recently developed (see [D4]) a list of seven axioms for a closure operation on modules that will induce the existence of a balanced big C-M R-module over a complete local domain R, independent of characteristic:

(1) $N_M^\natural$ is a submodule of M containing N.
(2) $(N_M^\natural)_M^\natural = N_M^\natural$; *i.e.*, the $\natural$-closure of N in M is closed in M.
(3) If $N \subseteq M \subseteq W$, then $N_W^\natural \subseteq M_W^\natural$.
(4) Let $f : M \to W$ be a homomorphism. Then $f(N_M^\natural) \subseteq f(N)_W^\natural$.
(5) If $N_M^\natural = N$, then $0_{M/N}^\natural = 0$.
(6) The ideals $\mathfrak{m}$ and 0 are $\natural$-closed; *i.e.*, $\mathfrak{m}_R^\natural = \mathfrak{m}$ and $0_R^\natural = 0$.
(7) Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters for R, and let $J = (x_1, \ldots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \to R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then
$$(Rv)_M^\natural \cap \ker f \subseteq (Jv)_M^\natural.$$

Conversely, given the existence of a big C-M R-module B, the operation given by
$$N_M^\natural := \{u \in M \,|\, b \otimes u \in \mathrm{Im}(B \otimes N \to B \otimes M) \text{ for all } b \in B\}$$
produces a closure operation that satisfies the seven axioms. It can be seen that tight closure (in positive characteristic or equal characteristic zero) also satisfies these axioms. Plus closure satisfies all seven axioms while Frobenius closure satisfies only the first six. Solid closure is much more mysterious (except in positive characteristic where it is equal to tight closure).

Such axioms as given above do not necessarily imply that there exists a closure operation for a particular class of rings. For example, for rings of mixed characteristic, whether there exist big C-M modules or algebras is an outstanding open question, and no closure operation is known to imply their existence in general. If, however, these axioms may help one construct such a closure operation.

My next step is to study axioms for big C-M R-algebras.

Question 3.1. What axioms should a closure operation satisfy in order to induce a big C-M algebra over a complete local domain R?

The results about seeds described above show that having a larger family of big C-M algebras also gives a “good” closure operation for complete local domains of positive characteristic. Finding axioms for such an existence of big C-M algebras will also be interesting.

4. Solid, Phantom, and Big Cohen-Macaulay Algebras

Another topic of interest for me has been the study of *solid* algebras. These were first defined by Hochster in [Ho] in his attempts to find an alternate definition of tight closure that would also yield a tight closure theory that did not depend upon characteristic. Given a domain R, an R-module M is *solid* if there exists a nonzero R-linear map $M \to R$. If $S = M$ is an R-algebra, then S is a solid algebra over R. Hochster also defined an operation called *solid closure* using this concept. If R is a complete local domain (the definition is more general but is simplest in this case), then the *solid closure* of I in R is
$$I_R^\bigstar := \{u \in R \,|\, u \in IS \text{ for some solid algebra } S\}.$$
When R contains a field of positive characteristic, Hochster has shown that solid closure is exactly tight closure (see [Ho]). This result motivates one to study solid algebras to try to gain better insight into tight closure.

In [HH3], Hochster and Huneke defined the notion of a *phantom extension* for finitely generated R-modules when R contains a field of positive characteristic. For a domain R, a map of R-modules $N \to M$ is a *phantom extension* if there exists $c \neq 0$ such that for all $e \gg 0$, there exists a map $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$, where $\mathbf{F}^e$ is the e^{th} iterated Frobenius functor.

Extending this definition to all R-modules, I have defined an R-algebra S to be *phantom* over R if the structure map $R \to S$ is a phantom extension. I have shown that if R is a complete local domain, then an R-algebra is solid if and only if it is phantom if and only if it is a direct limit of finitely generated R-modules M_α with $R \to M_\alpha$ a phantom extension, where the map is the restriction of the structure map $R \to S$. See [D1].

Hochster has shown that all R-algebras that map to a big C-M R-algebra are solid over R (see [Ho]). Hochster and Huneke gave an example in [HH5] showing that in equal characteristic, there are solid algebras that are not seed algebras.

I am still interested in studying phantom extensions, however, as they been useful in developing the axioms of the previous section and may be useful in developing axioms for big C-M algebras.

5. Tight Closure and Plus Closure

For a very long time, the most outstanding open problem in the theory of tight closure was whether or not tight closure commutes with localization, that is, if U is a multiplicative set in R and $S = U^{-1}R$, does

$$I^*_R S = (IS)^*_S?$$

This question has an affirmative answer if it is the case that

$$I^* = IR^+ \cap R,$$

when R is an excellent local domain. (The ideal $IR^+ \cap R$ is called the *plus closure* of I.) This follows because R^+ localizes nicely, that is, $(R^+)_P \cong (R_P)^+$, for a prime ideal P of R. Perhaps the best partial answer to this question comes via the work of K.E. Smith ([Sm]) which shows that if an ideal I is generated by part of an s.o.p. in an excellent local domain, then $I^* = IR^+ \cap R$. There is also a similar graded result (see [Sm2]) that equates tight closure and the contracted expansion of an ideal from $R^{+\mathrm{GR}}$.

Later, H. Brenner showed in [Br] and [Br2] that if R is the two-dimensional homogeneous coordinate ring of an elliptic curve or a two-dimensional $\mathbb{N}$-graded domain of finite type over the algebraic closure of a finite field (respectively), then $I^* = IR^+ \cap R = IR^{+\mathrm{GR}} \cap R$ for all homogeneous ideals primary to the unique homogeneous maximal ideal. Analogous statements for modules can be derived by essentially the same techniques.

Using a graded generalization of the Briançon-Skoda Theorem, I have been able to show that if R is a standard graded K-algebra domain of positive characteristic with K algebraically closed such that $N^*_M = N^+_M = N^{+\mathrm{GR}}_M$ for all finitely generated R-modules, where M/N is graded and coprimary to the homogeneous maximal ideal, then the same is true without requiring that M/N be graded. In particular, when we apply this result to the work of Brenner, then $I^* = IR^+ \cap R = IR^{+\mathrm{GR}} \cap R$ for all ideals primary to the unique homogeneous maximal ideal, whether they are homogeneous or not, where R is the homogeneous coordinate ring of an elliptic curve or a two-dimensional $\mathbb{N}$-graded domain of finite type over the algebraic closure of a finite field. See [D1] or [D2].

For a while I had hoped to push this result further and strengthen the evidence that tight closure and plus closure are equal, but Brenner and Monsky have laid this hope to rest as they have produced an example where the two closure operations are not equal. (See [BM].)

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