TIGHT CLOSURE OF FINITE LENGTH MODULES IN GRADED RINGS

GEOFFREY D. DIETZ

Abstract. In this article, we look at how the equivalence of tight closure and plus closure (or Frobenius closure) in the homogeneous m-coprimary case implies the same closure equivalence in the non-homogeneous m-coprimary case in standard graded rings. Although our result does not depend upon dimension, the primary application is based on results known in dimension 2 due to the recent work of H. Brenner. We also demonstrate a connection between tight closure and the m-adic closure of modules extended to R^+ or R^∞. We finally show that unlike the Noetherian case, the injective hull of the residue field over R^+ or R^∞ contains elements that are not killed by any power of the maximal ideal of R. This fact presents an obstruction to one possible method of extending our main result to all modules.

1. Introduction

Tight closure theory, since its introduction by M. Hochster and C. Huneke in the late 1980s, has been an important method of working in positive characteristic. The *tight closure* of an ideal is defined to be

$$I^* := \{x \in R \,|\, \exists c \in R^\circ \text{ such that } cx^{p^e} \in I^{[p^e]} \quad \forall e \gg 0\}$$

where p is the characteristic of the ring, R° is the complement in R of the minimal primes of R, and $I^{[p^e]}$ is the ideal generated by all p^eth powers of the elements of I. Some of the success of tight closure has been due to its ability to tie together ideas that were previously not known to be connected, generalize theorems, and simplify proofs. Some of the many examples include generalizations of the Briançon-Skoda Theorem and the Hochster-Roberts Theorem. See [HH1] for a general introduction.

Deciding whether tight closure computations commute with localization or not remains a very elusive goal. See [AHH] for results on this problem. Closely related to the localization issue is the question of whether the tight closure of an ideal I in a positive characteristic domain R is simply the contracted-expansion of I to R^+, the integral closure of R in an algebraic closure of its fraction field, i.e., does $I^* = IR^+ \cap R$? (The ideal $IR^+ \cap R$ is called the *plus closure* of I.) Since the plus closure operation can easily be shown to commute with localization, one could solve the localization problem by proving the above equality.

In [Sm1], K.E. Smith made a tremendous contribution to this problem by proving that $I^* = IR^+ \cap R$ for ideals generated by partial systems of parameters in excellent local domains of positive characteristic. Smith also showed that tight closure and

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plus closure are equal for ideals generated by part of a homogeneous system of parameters in an $\mathbb{N}$-graded domain of positive characteristic. See [Sm2] or Theorem 2.8 here.

Recently H. Brenner made a major breakthrough on this problem when he showed that tight closure and plus closure are equivalent for homogeneous ideals in certain 2-dimensional graded rings. See [Br2] and [Br3] and Theorems 2.9, 2.10, and 2.11 here. Brenner's work relies on a correspondence between tight closure and cohomological behavior of bundles on projective varieties (see [Br1]). Although it is not made explicit in his work, the same methods show the equivalence for finitely generated graded modules over the same class of rings. See Section 4 for details.

Inspired by Brenner's work, we have studied how one can obtain an equivalence of tight closure and plus closure for more general ideals and modules given that one has the equivalence for homogeneous ideals and modules. Our main result is the following theorem.

Theorem 3.5. *Let (R, m) be a standard graded K-algebra (see Section 3) of characteristic $p > 0$. Suppose that R is a domain, and K is algebraically closed. If $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$ for all finitely generated graded R-modules $N \subseteq M$ such that M/N is m-coprimary, then the same is true for* all *finitely generated modules $N \subseteq M$ such that M/N is m-coprimary.*

As a result, we can apply our theorem to the cases where Brenner's work is valid to increase the class of ideals and modules where tight closure equals plus closure.

Unlike the work of Brenner, our methods are entirely algebraic and rely on injective modules over a graded subring, $R^{+\mathrm{GR}}$, of R^+. This led us to the study of injective hulls over $R^{+\mathrm{GR}}$ and R^∞ in an attempt to extend our result beyond the m-coprimary case. We present a submodule of the injective hull $E_{A^\infty}(K)$, where A is either a polynomial ring or a formal power series ring, that we use to show that the injective hulls $E_{R^{+\mathrm{GR}}}(K)$, $E_{R^+}(K)$, and $E_{R^\infty}(K)$ behave far differently from the Noetherian case, as these modules contain elements that are not killed by any power of the maximal ideal of R. See Theorems 5.10 and 5.11. As a consequence we have not been able to use these injective modules to extend Theorem 3.5 to general modules.

2. Notation and Background

Before we state our results, we survey the theory that forms the foundation and provides the motivation for our work. All rings throughout are commutative with identity and are Noetherian unless noted otherwise. All modules are unital.

2.1. **The Frobenius Endomorphism.** We will always let p denote a positive prime number, and q will denote p^e, a power of p. Every characteristic p ring R comes equipped with a *Frobenius endomorphism* $F_R : R \to R$, which maps $r \mapsto r^p$. Composing this map with itself we obtain $F^e_R : R \to R$, which map $r \mapsto r^q$. Closely associated to these maps are the *Peskine-Szpiro* (or *Frobenius*) *functors* $\mathbf{F}^e_R$. If we let S denote the ring R viewed as an R-module via the e^{th}-iterated Frobenius endomorphism, then $\mathbf{F}^e_R$ is the covariant functor $S \otimes_R -$ which takes R-modules to S-modules and so takes R-modules to R-modules since $S = R$ as a ring. Specifically, if $R^m \to R^n$ is a map of free R-modules given by the matrix (r_{ij}), then we may apply $\mathbf{F}^e_R$ to this map to obtain a map between the same R-modules given by the

matrix (r_{ij}^q). For cyclic modules R/I, $\mathbf{F}_R^e(R/I) = R/I^{[q]}$, where

$$I^{[q]} := (a^q \,|\, a \in I)R$$

is the q^{th} *Frobenius power* of the ideal I. If the ideal I is finitely generated, then $I^{[q]}$ is also the ideal generated by the q^{th} powers of a finite generating set for I. In a similar manner, for modules $N \subseteq M$,

$$N_M^{[q]} := \mathrm{Im}(\mathbf{F}_R^e(N) \to \mathbf{F}_R^e(M)),$$

and we will denote the image of $u \in N$ inside of $N_M^{[q]}$ by u^q.

When R is reduced, we define $R^{1/q}$ to be the ring obtained by adjoining to R all q^{th} roots of elements in R. In this setting, the inclusion $R \hookrightarrow R^{1/q}$ is isomorphic to the inclusion $F_R^e : R \hookrightarrow R$, identifying $R^{1/q}$ with R via the isomorphism $r^{1/q} \mapsto r$. Therefore the Peskine-Szpiro functor $\mathbf{F}_R^e$ is isomorphic to $R^{1/q} \otimes_R -$ after identifying $R^{1/q}$ with R, and the Frobenius power $I^{[q]}$ can be identified in the same sense with the extension $IR^{1/q}$. We denote by R^∞ the *perfect closure* of R. The ring R^∞ is constructed by adjoining all q^{th} roots to R, for all q. In general, R^∞ is not a Noetherian ring.

Definition 2.1. For a Noetherian ring R of positive characteristic p and finitely generated R-modules $N \subseteq M$, the *Frobenius closure* of N in M is the submodule

$$N_M^F := \{u \in M \,|\, u^q \in N_M^{[q]}, \text{ for some } q\}.$$

Lemma 2.2. *If R is a reduced Noetherian ring of characteristic $p > 0$, then for finitely generated R-modules $N \subseteq M$, the following are equivalent:*

(i) $u \in N_M^F$.
(ii) $1 \otimes u \in \mathrm{Im}(R^{1/q} \otimes N \to R^{1/q} \otimes M)$, *for some* q.
(iii) $1 \otimes u \in \mathrm{Im}(R^\infty \otimes N \to R^\infty \otimes M)$.

In the case of ideals, $I^F = \bigcup_q IR^{1/q} \cap R = IR^\infty \cap R$.

2.2. **Tight Closure.** Let the complement in R of the set of minimal primes be denoted by R°.

Definition 2.3. For a Noetherian ring R of characteristic $p > 0$ and finitely generated modules $N \subseteq M$, the *tight closure* N_M^* of N in M is

$$N_M^* := \{u \in M \,|\, cu^q \in N_M^{[q]} \text{ for all } q \gg 1, \text{ for some } c \in R^\circ\}.$$

In the case that $M = R$ and $N = I$, $u \in I^*$ if and only if there exists $c \in R^\circ$ such that $cu^q \in I^{[q]}$, for all $q \gg 1$.

Using the existence of *test elements* (see [HH1, HH3]), we have some useful characterizations of tight closure using $R^{1/q}$ and R^∞.

Lemma 2.4. *Let $N \subseteq M$ be finitely generated R-modules, where R is reduced of positive characteristic p and has a test element. Then the following are equivalent:*

(i) $u \in N_M^*$.
(ii) $c^{1/q} \otimes u \in \mathrm{Im}(R^{1/q} \otimes N \to R^{1/q} \otimes M)$, *for all $q \gg 1$ and some $c \in R^\circ$.*
(iii) $c^{1/q} \otimes u \in \mathrm{Im}(R^\infty \otimes N \to R^\infty \otimes M)$, *for some (or every) test element c and all $q \geq 1$.*

2.3. **Plus Closure.** Let R be a domain, and let R^+ denote the integral closure of R in an algebraic closure of its fraction field. R^+ is called the *absolute integral closure of* R and is not Noetherian in general. This is an important ring due to the remarkable result of Hochster and Huneke.

Theorem 2.5 (Theorem 5.15, [HH2])**.** *Let R be an excellent local domain of positive characteristic. Then R^+ is a big Cohen-Macaulay R-algebra, i.e., every system of parameters of R is a regular sequence on R^+.*

Hochster and Huneke also provide a graded version of the above theorem. If R is an $\mathbb{N}$-graded domain, then $R^{+\mathrm{GR}}$ denotes a maximal direct limit of module-finite, $\mathbb{Q}_{\geq 0}$-graded extension domains of R. For the construction and properties of this ring, see [HH2, Lemma 4.1]. There is also an $\mathbb{N}$-graded direct summand of $R^{+\mathrm{GR}}$, which is denoted $R^{+\mathrm{gr}}$. Neither of these rings is Noetherian in general.

Theorem 2.6 (Theorem 5.15, [HH2])**.** *If R is an $\mathbb{N}$-graded domain of positive characteristic with $R_0 = K$ and R a finitely generated K-algebra, then $R^{+\mathrm{GR}}$ and $R^{+\mathrm{gr}}$ are both graded big Cohen-Macaulay R-algebras in the sense that every homogeneous system of parameters of R is a regular sequence on $R^{+\mathrm{GR}}$ and $R^{+\mathrm{gr}}$.*

There is a closure operation associated to each of the rings R^+, $R^{+\mathrm{GR}}$, and $R^{+\mathrm{gr}}$. Since $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$ as an $R^{+\mathrm{gr}}$-module, the latter two rings yield equivalent closure operations.

Definition 2.7. Given an excellent, local (resp., Noetherian, $\mathbb{N}$-graded) domain R of positive characteristic, let $S = R^+$ (resp., $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$), and let $N \subseteq M$ be R-modules. The *plus closure* N^+_M (resp., *graded-plus closure* $N^{+\mathrm{gr}}_M$) of N in M is

$$\{u \in M \mid 1 \otimes u \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)\}.$$

If $M = R$ and $N = I$, then I^+ (resp., $I^{+\mathrm{gr}}$) equals $IS \cap R$.

It is straightforward to show that $N^F_M \subseteq N^+_M \subseteq N^*_M$, for all finitely generated modules, and that $N^F_M \subseteq N^{+\mathrm{gr}}_M \subseteq N^+_M \subseteq N^*_M$ in the graded case. As mentioned earlier, it is possible that tight closure in positive characteristic is just plus closure. Some of the most important results in this direction come from K.E. Smith.

Theorem 2.8 ([Sm1], [Sm2])**.** *Let R be an excellent local (resp., $\mathbb{N}$-graded) Noetherian domain of characteristic $p > 0$ (with R_0 a field), and let $I = (x_1, \ldots, x_k)$ be an ideal generated by part of a (homogeneous) system of parameters. Then $I^* = I^+$ (resp., $I^* = I^{+\mathrm{gr}} = I^+$).*

Recently, H. Brenner has developed results in dimension 2 that show that tight closure and graded-plus closure are equivalent for homogeneous ideals in certain graded rings.

Theorem 2.9 (Theorem 4.3, [Br2])**.** *Let K be an algebraically closed field of positive characteristic, and let R be the homogeneous coordinate ring of an elliptic curve (i.e., R is a standard graded normal K-algebra of dimension 2 with $\dim_K [H^2_m(R)]_0 = 1$, where m is the homogeneous maximal ideal of R). Let I be an m-primary graded ideal in R. Then $I^{+\mathrm{gr}} = I^+ = I^*$.*

For example, the result above applies when $R = K[x, y, z]/(F)$ is normal, where F is homogeneous of degree 3.

Theorem 2.10 (Theorem 4.2, [Br3]). *Let K be the algebraic closure of a finite field. Let R denote an $\mathbb{N}$-graded 2-dimensional domain of finite type over K. Then for every homogeneous ideal I, we have $I^{+\mathrm{gr}} = I^{+} = I^{*}$.*

In the case of an elliptic curve with Hasse invariant 0 (see [Ha, pp. 332–335]), Brenner showed that tight closure is the same as Frobenius closure.

Theorem 2.11 (Remark 4.4, [Br2]). *If R is the homogeneous coordinate ring of an elliptic curve of positive characteristic p with Hasse invariant 0 defined over an algebraically closed field, then $I^{*} = I^{F}$ for all m-primary graded ideals of R, where m is the homogeneous maximal ideal of R.*

3. New Cases Where Tight Closure is Plus Closure

Before proving our main result, Theorem 3.5, we need to establish some lemmas and notation. If S is any $\mathbb{Q}$-graded (not necessarily Noetherian) ring, then for any $n \in \mathbb{Q}$ let $S_{\geq n} = \bigoplus_{i \geq n} S_i$. Similarly define $S_{>n}$. We will say that an $\mathbb{N}$-graded ring R is a *standard graded R_0-algebra* if R is finitely generated over R_0 by elements of degree 1. For the rest of the section, let $m = \bigoplus_{i>0} R_i$, the homogeneous maximal ideal of R.

Lemma 3.1. *Let R be a reduced standard graded K-algebra of positive characteristic p, and let $S = R^{\infty}$. Then there exists $c \in \mathbb{N}$ such that $S_{\geq n+c} \subseteq m^n S$ for any $n \geq 1$. As a consequence, $[S/m^n S]_j = 0$ for all $j \gg 0$.*

Proof. Let m be generated by $x_1, \ldots, x_\mu$, each of degree 1. Put $c = \mu - 1$ (if $\mu = 0$, i.e., $R = K$, put $c = 0$). Since $S_{>0} = \cup_q m^{1/q}$, if $f \in S$ is homogeneous of degree at least $n+c$, then f is a sum of terms $d x_1^{\alpha_1} \cdots x_\mu^{\alpha_\mu}$ such that $d \in S_0$ and $\sum \alpha_i \geq n+c$. If we write $\alpha_i = [\alpha_i] + r_i$, where $0 \leq r_i < 1$ for all i, then

$$\sum [\alpha_i] = \sum \alpha_i - \sum r_i \geq n + c - \sum r_i > n + c - \mu.$$

Therefore, $\sum [\alpha_i] \geq n + c - \mu + 1 = n$, and so $f \in m^n S$. The second claim now follows since $j \geq n + c$. □

To prove a similar result for $R^{+\mathrm{GR}}$, we will need a graded-plus closure version of the Briançon-Skoda Theorem. The original tight closure generalization can be found in [HH1, Theorem 5.4]. Hochster and Huneke also strengthened this result to a version for plus closure in [HH4, Theorem 7.1]. We will adapt their proof to obtain a graded-plus closure version of the Briançon-Skoda Theorem.

Theorem 3.2. *Let R be a positively graded Noetherian domain of positive characteristic. Let I be a homogeneous ideal generated by at most d homogeneous elements, let $k \in \mathbb{N}$, and let $u \in \overline{I^{d+k}}$ with u homogeneous. Then $u \in I^{k+1} S \cap R$, where $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$.*

Proof. We will first use R, I, and u to construct a triple (A, J, v) and a degree-preserving map to the triple (R, I, u) such that A is a positively graded Noetherian domain, J is a homogeneous ideal of A with at most d generators, $v \in \overline{J^{d+k}}$ is homogeneous in A, $I = JR$, and $v \mapsto u$. We will directly prove the theorem holds for (A, J, v) and then show that this case implies the result for (R, I, u).

Since $u \in R$ is integral over I^{d+k} and homogeneous, u satisfies a homogeneous monic polynomial

$$z^n + r_1 z^{n-1} + \cdots + r_n = 0,$$

where $\deg z = \deg u$, $\deg r_j = j \deg u$, $r_j \in (I^{d+k})^j$, and (without loss of generality) $r_n \neq 0$. Each r_j can be written as a homogeneous R-linear combination of monomials $a_1^{\nu_1} \cdots a_d^{\nu_d}$ in the generators $a_1, \ldots, a_d$ of I, where $\nu_1 + \cdots + \nu_d = (d+k)j$. Thus, the coefficient of the monomial $a_1^{\nu_1} \cdots a_d^{\nu_d}$ is zero or has

$$\text{degree } = \deg r_j - (\nu_1 \deg a_1 + \cdots + \nu_d \deg a_d)$$

since R is positively graded. Without loss of generality, we may order the generators of I so that $\deg a_1 \leq \cdots \leq \deg a_d$. Then $r_n \neq 0$ implies that $\deg a_1^{(d+k)n} \leq \deg r_n$. If not, then $\deg r_n < \nu_1 \deg a_1 + \cdots + \nu_d \deg a_d$, for all ν_i such that $\nu_1 + \cdots + \nu_d = (d+k)n$, and so the coefficient of every monomial in the expansion of r_n must be zero, a contradiction.

Let $x_1, \ldots, x_d$ be indeterminates over $K = \mathbb{Z}/p\mathbb{Z}$ with $\deg x_i = \deg a_i$. For every monomial $\mu = x_1^{\nu_1} \cdots x_d^{\nu_d}$, where $\nu_1 + \cdots + \nu_d = (d+k)j$ for $1 \leq j \leq n$, let y_μ be an indeterminate with $\deg y_\mu = \deg r_j - (\nu_1 \deg x_1 + \cdots + \nu_d \deg x_d)$. Let

$$F(\mathbf{x}, \mathbf{y}, z) = z^n + \sum_{j=1}^{n} \left(\sum_{\mu \in C_j} y_\mu \mu \right) z^{n-j},$$

where $\mathbf{x} = x_1, \ldots, x_d$, $\mathbf{y} = \{y_\mu \mid \deg y_\mu \geq 0\}$, and

$$C_j = \{\mu = x_1^{\nu_1} \cdots x_d^{\nu_d} \mid \nu_1 + \cdots + \nu_d = (d+k)j\}.$$

Then F is homogeneous of degree $n \deg z = n \deg u$ as

$$\deg(y_\mu \mu) z^{n-j} = \deg r_j + (n-j) \deg z = j \deg u + n \deg z - j \deg z = n \deg z.$$

Therefore, $K[\mathbf{x}, \mathbf{y}, z]$ is a positively graded Noetherian ring, and the homomorphism $K[\mathbf{x}, \mathbf{y}] \to R$, given by $x_i \mapsto a_i$ and mapping y_μ to the coefficient of $a_1^{\nu_1} \cdots a_d^{\nu_d}$, is degree-preserving. Moreover, the composite map $K[\mathbf{x}, \mathbf{y}, z] \to R[z] \to R$, where $z \mapsto u$, sends $F(\mathbf{x}, \mathbf{y}, z) \mapsto z^n + r_1 z^{n-1} + \cdots + r_n \mapsto 0$. (Since R is positively graded, the coefficient of $a_1^{\nu_1} \cdots a_d^{\nu_d}$ is 0 if $\deg r_j < \nu_1 \deg a_1 + \cdots + \nu_d \deg a_d$ so that we did not need a $y_\mu \mu$ term in F when $\deg y_\mu < 0$.)

Put $A := K[\mathbf{x}, \mathbf{y}, z]/F(\mathbf{x}, \mathbf{y}, z)$, $J := (\mathbf{x})A$, and $v := z$ in A. Then A is a positively graded Noetherian ring of positive characteristic, J is a homogeneous ideal generated by at most d homogeneous elements, and v is homogeneous and in $\overline{J^{d+k}}$. It is clear from the construction of A that $A \to R$ is a degree-preserving map, that $I = JR$, and that $v \mapsto u$ under the map. To see that A is also a domain, we will show that F is irreducible. Indeed, let $N = (d+k)n$, and let μ be the monomial x_1^N that occurs when $j = n$ in the summation for F. As we noted earlier, $r_n \neq 0$ implies that

$$\deg y_\mu = \deg r_n - N \deg x_1 = \deg r_n - N \deg a_1 \geq 0,$$

and so F is linear in y_μ with coefficient x_1^N for y_μ and a relatively prime constant term containing z^n.

We next show that the theorem holds for the triple (A, J, v) constructed above. As $A^{+\mathrm{gr}}$ is a direct summand of $A^{+\mathrm{GR}}$, it is enough to show that $v \in JA^{+\mathrm{GR}} \cap A$. Since A is a positively graded, finitely generated K-algebra, we may regrade if necessary so that it is $\mathbb{N}$-graded without changing $A^{+\mathrm{GR}}$. Since $A/(\mathbf{x}) \cong K[\mathbf{y}, z]/z^n$, the sequence $x_1, \ldots, x_d$ forms part of a homogeneous system of parameters. We can now apply Theorem 2.8 to the ring A and ideal J to see that $J^* = JA^{+\mathrm{GR}} \cap A$. By the Generalized Briançon-Skoda Theorem ([HH1, Theorem 5.4]), the theorem holds for (A, J, v).

We finally claim that the theorem holds for the original triple (R, I, u). (We again only need to show the $R^{+\mathrm{GR}}$ case.) Since the map $A \to R$ extends to $A^+ \to R^+$, we can restrict this map to obtain $A^{+\mathrm{GR}} \to R^{+\mathrm{GR}}$ (The homogeneous monic equation satisfied by an element a of $A^{+\mathrm{GR}}$ maps to a homogeneous monic equation over R satisfied by the image of a.) Therefore, $u \in I^{k+1}R^{+\mathrm{GR}}$ as $v \in J^{k+1}A^{+\mathrm{GR}}$, $v \mapsto u$, and $JR = I$. □

Lemma 3.3. *Let R be a standard graded K-algebra domain of characteristic $p > 0$, and let $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$. Then there exists $c \in \mathbb{N}$ such that $S_{\geq n+c} \subseteq m^n S$, for any $n \geq 1$. Moreover, $[S/m^n S]_j = 0$ for all $j \gg 0$.*

Proof. Let m be generated by μ elements. Let $c = \mu - 1$ (if $\mu = 0$, let $c = 0$), and let $f \in S$ be homogeneous of degree $D \geq n+c$. Then f satisfies a monic polynomial equation $f^t + r_1 f^{t-1} + \cdots + r_t = 0$ such that r_i is homogeneous of degree iD in R or $r_i = 0$ if $iD \notin \mathbb{N}$. Therefore, $r_i \in m^{i(n+c)} = (m^{n+c})^i$ for all i as m is generated in degree 1. Since $f \in S$, there exists a positively graded module-finite extension domain T of R such that $f \in T$. Thus, $f \in \overline{(mT)^{n+c}} = \overline{(mT)^{\mu+n-1}}$. By Theorem 3.2, $f \in m^n T^{+\mathrm{GR}}$, but $T^{+\mathrm{GR}} = R^{+\mathrm{GR}}$, and so $f \in m^n R^{+\mathrm{GR}}$. Since $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$, we also have $f \in m^n R^{+\mathrm{gr}}$. The second claim follows using $j \geq n + c$. □

Our main result will depend upon showing that $\mathrm{Hom}_K(S/m^n S, K)$ is $\mathbb{Z}$-graded as an R-module when S is $R^{+\mathrm{GR}}$, $R^{+\mathrm{gr}}$, or R^∞.

Proposition 3.4. *Let R be a standard graded K-algebra of characteristic $p > 0$. Suppose R is reduced (respectively, a domain). Let $S = R^\infty$ (resp., $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$). Then for any $n \geq 1$, $\mathrm{Hom}_K(S/m^n S, K)$ is a $\mathbb{Z}$-graded R-module.*

Proof. S has a natural $\mathbb{N}[1/p]$-grading (resp., $\mathbb{Q}_{\geq 0}$-grading or $\mathbb{N}$-grading) induced by the grading on R. Thus, $S/m^n S$ is also graded as $m^n S$ is a homogeneous ideal. Let W_j be the K-span of all homogeneous elements of degree δ such that $j - 1 < \delta \leq j$. This gives $S/m^n S$ an $\mathbb{N}$-grading as an R-module, where $W_j = 0$ for all $j < 0$ and $j \gg 0$ by Lemma 3.1 (resp., Lemma 3.3).

In $\mathrm{Hom}_K(S/m^n S, K)$, let V_{-j} be the K-span of all functionals ϕ such that $\phi(W_i)$ is not 0 if and only if $i = j$. If $r \in R$ is homogeneous of degree d, and $\phi \in V_{-j}$, then $r\phi(W_i) = \phi(rW_i) \subseteq \phi(W_{i+d})$ which is nonzero if and only if $i = -d + j$. Thus, $R_d V_{-j} \subseteq V_{-j+d}$. It is clear that the intersection of any V_{-j} with the sum of the others is trivial and that $\sum_j V_{-j} \subseteq \mathrm{Hom}_K(S/m^n S, K)$. Now, if $\psi \in \mathrm{Hom}_K(S/m^n S, K)$, and s has homogeneous components s_i, then let $\psi_{-j}(s) = \psi(s_j)$ so that $\psi_{-j} \in V_{-j}$. Then $\psi = \sum_j \psi_{-j}$, where the sum is finite because W_i is nonzero for only finitely many integers. Therefore the V_{-j} give a $\mathbb{Z}$-grading on $\mathrm{Hom}_K(S/m^n S, K)$ as an R-module. □

We are now ready to present the main result. The method of the proof will be to show that if M is an m-coprimary module containing an element $u \in 0_M^* \setminus 0_M^{+\mathrm{GR}}$, then M can be mapped to a finitely generated graded m-coprimary R-module where the image of u is not in the plus closure of 0 in this new module.

Theorem 3.5. *Let R be a standard graded K-algebra of characteristic $p > 0$. Suppose that R is reduced (respectively, a domain), and K is perfect (resp., algebraically closed). If $N_M^* = N_M^F$ (resp., $N_M^* = N_M^+ = N_M^{+\mathrm{gr}}$) for all finitely generated graded*

R-modules $N \subseteq M$ such that M/N is m-coprimary, then the same is true for all finitely generated modules $N \subseteq M$ such that M/N is m-coprimary.

Proof. Let $S = R^{\infty}$ (resp., $S = R^{+\mathrm{GR}}$). It suffices to show that $0^*_M \subseteq 0^F_M$ (resp., $0^*_M \subseteq 0^{+\mathrm{gr}}_M$) when M is m-coprimary. Suppose that $u \in 0^*_M \setminus 0^F_M$ (resp., $u \in 0^*_M \setminus 0^{+\mathrm{gr}}_M$). Since $u \notin 0^F_M$ (resp., $u \notin 0^{+\mathrm{gr}}_M$), $1 \otimes u \neq 0$ via the natural map $\phi : M \to S \otimes M$. This implies that there is a surjection of the cyclic S-module $S(1 \otimes u)$ onto K sending $1 \otimes u$ to $1 \in K$, since the residue field of S is K.

Since $\mathrm{Hom}_K(S, K)$ is an injective S-module and since we have a homomorphism $K \to \mathrm{Hom}_K(S, K)$ that sends 1 to the functional that takes $s \in S$ to s modulo m_S, there exists a homomorphism ψ as in the diagram below

$$\begin{array}{ccccccc} M & \xrightarrow{\phi} & S \otimes M & & & & \\ \uparrow & & \uparrow & \overset{\psi}{\searrow} & & & \\ Ru & \longrightarrow & S(1 \otimes u) & \longrightarrow & K & \longrightarrow & \mathrm{Hom}_K(S, K) \end{array}$$

such that $\psi \circ \phi(u) \neq 0$. Since M is m-coprimary, there exists an n such that $m^n M = 0$. Hence, the image of M under $\psi \circ \phi$ lies in the annihilator of m^n in $\mathrm{Hom}_K(S, K)$, which is isomorphic to $\mathrm{Hom}_K(S/m^n S, K)$.

By Proposition 3.4, $\mathrm{Hom}_K(S/m^n S, K)$ is a $\mathbb{Z}$-graded R-module. Let M' be the R-submodule of $\mathrm{Hom}_K(S/m^n S, K)$ generated by the homogeneous components of the images of the generators of M. Then M' is a finitely generated m-coprimary graded R-module.

Let $\tilde{u} = \psi \circ \phi(u)$, which we know is nonzero and in M'. As $u \in 0^*_M$, we also have that $\tilde{u} \in 0^*_{M'}$. By our hypothesis, $\tilde{u} \in 0^F_{M'}$ (resp., $\tilde{u} \in 0^{+\mathrm{gr}}_{M'}$) since M' is graded and m-coprimary. Therefore, $1 \otimes \tilde{u} = 0$ in $S \otimes M'$. Since $\mathrm{Hom}_K(S/m^n S, K)$ is an S-module, the inclusion map $M' \hookrightarrow \mathrm{Hom}_K(S/m^n S, K)$ factors through the map $M' \to S \otimes M'$, by the universal property of base change. Thus, the fact that $1 \otimes \tilde{u} = 0$ in $S \otimes M'$ implies that the image of $\tilde{u}$ is 0 in $\mathrm{Hom}_K(S/m^n S, K)$, a contradiction. □

4. An Application in Dimension 2

In [Br2] and [Br3] (Theorems 2.9 and 2.10 here), Brenner shows cases where the tight closure of a primary homogeneous ideal is the same as its graded-plus closure. Brenner has observed in correspondence that it is straightforward to generalize these results to include finitely generated m-coprimary R-modules. The argument is lengthy, like the one for ideals, but the changes are routine. (The main idea is to replace the syzygy bundle constructed from homogeneous generators of an m-primary ideal with a syzygy bundle constructed from homogeneous generators of an m-coprimary submodule N of a graded module M. Once one has made the necessary alterations to [Br1, Section 3], all of the relevant proofs in [Br2] and [Br3] follow seamlessly as they only rely on the aforementioned results and theorems whose hypotheses only require locally free sheaves of arbitrary rank, which we obtain in both the ideal and module cases.) We state this generalization as the following theorem.

Theorem 4.1 (H. Brenner)**.** *Let R be a positive characteristic ring. Further, let R be the homogeneous coordinate ring of an elliptic curve over an algebraically closed*

field K, or let R be any 2-dimensional standard graded K-algebra, where K is the algebraic closure of a finite field. Let $N \subseteq M$ be finitely generated graded R-modules such that M/N is m-coprimary, where m is the homogeneous maximal ideal of R. Then $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$.

This result together with Theorem 3.5 yields an extension of Theorem 4.1.

Corollary 4.2. *With R as above, $N^*_M = N^+_M = N^{+\mathrm{gr}}_M$ for all finitely generated R-modules such that M/N is m-coprimary.*

Further, if Proj R is an elliptic curve with Hasse invariant 0, then Brenner's Theorem 2.11 says that the tight closure of a primary homogeneous ideal is the same as its Frobenius closure. For example, this is the case for the cubical cone $R = K[x, y, z]/(x^3+y^3+z^3)$, when the characteristic of K is congruent to 2 (mod 3) (as implied by [Ha, Proposition 4.21]). Again, Brenner's result can be generalized to include finitely generated homogeneous modules $N \subseteq M$ with m-coprimary quotients. This fact can then be paired with Theorem 3.5 to give:

Corollary 4.3. *If R is the homogeneous coordinate ring of an elliptic curve of positive characteristic p with Hasse invariant 0 defined over an algebraically closed field, then $N^*_M = N^F_M$ for all finitely generated R-modules such that M/N is m-coprimary.*

For a Noetherian ring R with a maximal ideal m, given the equivalence of tight closure and plus closure (respectively, graded-plus closure or Frobenius closure) in the m-coprimary case, we can present a characterization of tight closure over R_m in terms of the m-adic closure of certain modules over R^+ (resp., $R^{+\mathrm{GR}}$, $R^{+\mathrm{gr}}$, or R^∞). We start with some general lemmas about tight closure.

Lemma 4.4. *Let (R, m) be a reduced local ring of positive characteristic p that has a test element. Let M be a finitely generated R-module. Then $u \in 0^*_M$ if and only if $u \in \bigcap_k (m^k M)^*_M$.*

Proof. Let c be a test element in R. Then $u \in 0^*_M$ if and only if $c^{1/q} \otimes u = 0$ in $R^{1/q} \otimes M$ for all q by Lemma 2.4. This holds if and only if

$$c^{1/q} \otimes u \in \bigcap_k m^k (R^{1/q} \otimes M)$$

for all q since $(R^{1/q}, m^{1/q})$ is also local, $R^{1/q} \otimes M$ is a finitely generated $R^{1/q}$-module, and the powers of $mR^{1/q}$ are cofinal with the powers of $m^{1/q}$. Since $m^k(R^{1/q} \otimes M) = \mathrm{Im}(R^{1/q} \otimes m^k M \to R^{1/q} \otimes M)$, the above occurs if and only if

$$c^{1/q} \otimes u \in \mathrm{Im}(R^{1/q} \otimes m^k M \to R^{1/q} \otimes M),$$

for all k and all q. Finally, since c is a test element, the previous holds if and only if $u \in (m^k M)^*_M$ for all k. □

Lemma 4.5. *Let R be a reduced ring, I an ideal, and $S = R^\infty$ (respectively, R is also a domain and $S = R^+$ or R is also a graded domain and $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$). Then $u \in (I^k M)^F_M$ (resp., $u \in (I^k M)^+_M$ or $u \in (I^k M)^{+\mathrm{gr}}_M$) for all k if and only if $1 \otimes u \in \bigcap_k I^k(S \otimes M)$.*

Proof. By definition, $u \in (I^k M)^F_M$ (resp., $u \in (I^k M)^+_M$ or $u \in (I^k M)^{+\mathrm{gr}}_M$) if and only if $1 \otimes u \in \mathrm{Im}(S \otimes I^k M \to S \otimes M)$. This holds if and only if $1 \otimes u \in I^k(S \otimes M)$.

Therefore, $u \in (I^k M)^F_M$ (resp., $u \in (I^k M)^+_M$ or $u \in (I^k M)^{+\mathrm{gr}}_M$) for all k if and only if $1 \otimes u \in \bigcap_k I^k(S \otimes M)$. □

We now give the promised result connecting tight closure in R_m and the m-adic closure in R^+ (resp., $R^{+\mathrm{GR}}$, $R^{+\mathrm{gr}}$, or R^∞).

Proposition 4.6. *Let R be a reduced ring of characteristic $p > 0$. Let m be a maximal ideal of R such that R_m has a test element (e.g., R_m is excellent). Let $S = R^\infty$ (resp., let R also be a domain and $S = R^+$ or let R be a graded domain and $S = R^{+\mathrm{GR}}$ or $S = R^{+\mathrm{gr}}$). Moreover, let R be such that Frobenius closure (resp., plus closure or graded-plus closure) equals tight closure for finitely generated modules with m-coprimary quotient.*

Then for any finitely generated $N \subseteq M$ and $u \in M$, we have $u/1 \in (N_m)^*_{M_m}$ if and only if $1 \otimes \overline{u}$ is in the m-adic closure of 0 in $S \otimes M/N$. For M free, we further note that $(N_m)^*_{M_m} \cap M = \bigcap_k (N + m^k M)S \cap M$.

Proof. Since $x \in (N_m)^*_{M_m}$ if and only if $\overline{x} \in 0^*_{M_m/N_m}$ and $M_m/N_m \cong (M/N)_m$, it is enough to show this for the case $N = 0$. By Lemma 4.4, $u/1 \in 0^*_{M_m}$ if and only if $u/1 \in \bigcap_k (m^k M_m)^*_{M_m}$. Since $M/m^k M$ is clearly m-coprimary, [HH1, Proposition 8.9] shows that the contraction of $(m^k M_m)^*_{M_m}$ to M is just $(m^k M)^*_M$ for all k. Hence $u/1 \in 0^*_{M_m}$ if and only if $u \in \bigcap_k (m^k M)^*_M$. By our hypothesis, this holds if and only if $u \in \bigcap_k (m^k M)^F_M$ (resp., $u \in \bigcap_k (m^k M)^+_M$ or $u \in \bigcap_k (m^k M)^{+\mathrm{gr}}_M$). Then Lemma 4.5 shows this is equivalent to $1 \otimes u \in \bigcap_k m^k(S \otimes M)$.

In the case that M is free, the above shows $u \in (N_m)^*_{M_m} \cap M$ if and only if $1 \otimes \overline{u} \in \bigcap_k m^k(S \otimes M/N)$, but $m^k(S \otimes M/N) \cong m^k(MS/NS)$ in this case. Further, $\overline{u} \in m^k(MS/NS)$ if and only if $u \in (N + m^k M)S$. □

5. Injective Hulls Over R^∞ and R^+

In this final section we study the injective hull of the residue field of R^∞, R^+, and $R^{+\mathrm{GR}}$, where R has positive characteristic and is a complete local domain or a standard graded K-algebra domain. We start by studying the injective hull, $E_{R^\infty}(K^\infty)$, where $R = K[[x_1, \ldots, x_n]]$ or $R = K[x_1, \ldots, x_n]$. In dimension $n \geq 2$, we show that there are elements of $E_{R^\infty}(K^\infty)$ that are not killed by any power of the maximal ideal of R. We then show that this result also holds for complete local domains and standard graded K-algebra domains in positive characteristic, and that it holds for the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$ as well. This latter result shows that we cannot extend Theorem 3.5 by making use of the injective hull of the residue field of $R^{+\mathrm{GR}}$ or R^∞ in the analogous way.

5.1. **The Regular Case.** In order to study the injective hull $E_{R^\infty}(K^\infty)$, where $R = K[[x_1, \ldots, x_n]]$ or $R = K[x_1, \ldots, x_n]$, we construct a submodule of formal sums such that the support has DCC. The supports will be subsets of $-\mathbb{N}[1/p]^n$, the set of n-tuples of nonpositive rational numbers whose denominators are powers of p. Throughout the rest of this section, we will use bold letters to stand for n-tuples of elements. We will place a partial ordering on n-tuples by comparing coordinate-wise, e.g., $\mathbf{a} > \mathbf{b}$ if and only if $a_i \geq b_i$, for all i, and $a_j > b_j$, for some j. We will define addition and subtraction of n-tuples as usual. If $\mathbf{a} \in \mathbb{Q}^n$, then $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$.

Definition 5.1. Let $R = K[[\mathbf{x}]]$ or $R = K[\mathbf{x}]$, where K is a field of positive characteristic p and $\dim R = n$. Given a formal sum $f = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{x}^{-\mathbf{a}}$, where $\mathbf{a} \in \mathbb{N}[1/p]^n$ and $c_{\mathbf{a}} \in K^\infty$, we will say that the *support* of f is the subset of $(-\mathbb{N}[1/p])^n$ given by

$$\operatorname{supp}(f) := \{-\mathbf{a} \,|\, c_{\mathbf{a}} \neq 0\}.$$

Using the same notation, we define the following set of formal sums

$$N := \{f = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{x}^{-\mathbf{a}} \,|\, \mathbf{a} \in \mathbb{N}[1/p],\, c_{\mathbf{a}} \in K^\infty, \text{ and } \operatorname{supp}(f) \text{ has DCC}\}.$$

Lemma 5.2. *Using the notation of Definition 5.1, N is an R^∞-module with formally defined multiplication.*

Proof. Let f_1, f_2 be in N, and let $\operatorname{supp}(f_i) = A_i$. Then $\operatorname{supp}(f_1 + f_2) \subseteq A_1 \cup A_2$. Since the union of two sets with DCC has DCC and a subset of a set with DCC also has DCC, $f_1 + f_2$ is in N. Now, let $s \in R^\infty$. Then $s \in R^{1/q}$, for some $q = p^e$, so that we can write

$$s = \sum_{\mathbf{b} \geq 0} d_{\mathbf{b}}\mathbf{x}^{\mathbf{b}/q},$$

where $\mathbf{b} \in \mathbb{N}^n$ and $d_{\mathbf{b}} \in K^\infty$. Put

$$f := \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{x}^{-\mathbf{a}} \in N.$$

Using formal multiplication, the coefficient of $\mathbf{x}^{-\mathbf{s}}$ in sf is

$$\sum_{-\mathbf{a}+\mathbf{b}/q=-\mathbf{s}} c_{\mathbf{a}}d_{\mathbf{b}}. \tag{5.3}$$

Notice that the coefficient of $\mathbf{x}^{-\mathbf{s}}$ is 0 if $-\mathbf{s} = -\mathbf{a} + \mathbf{b}/q > 0$ as $K^\infty = S/m_S$.

When $-\mathbf{s} \leq 0$, for sf to be well-defined, the summation (5.3) must consist of a finite sum of nonzero elements. In the polynomial case, this is clear. Otherwise, suppose that we have enumerated the terms contributing to the coefficient of $\mathbf{x}^{-\mathbf{s}}$ and that the set

$$\{k \in \mathbb{N} \,|\, -\mathbf{a}^{(k)} + \mathbf{b}^{(k)}/q = -\mathbf{s}, \text{ and } c_{\mathbf{a}}d_{\mathbf{b}} \neq 0\}$$

is infinite. If there are only finitely many distinct $\mathbf{b}^{(k)}$, then (5.3) is clearly a finite sum. We may then assume that there are infinitely many distinct $\mathbf{b}^{(k)}$ and thus assume that all of the $\mathbf{b}^{(k)}/q$ are distinct. Hence, we obtain an infinite chain of equalities

$$-\mathbf{a}^{(1)} + \mathbf{b}^{(1)}/q = -\mathbf{a}^{(2)} + \mathbf{b}^{(2)}/q = -\mathbf{a}^{(3)} + \mathbf{b}^{(3)}/q = \cdots.$$

Since the sets $\mathbb{N}/q$ and $\operatorname{supp}(f)$ have DCC, we may apply Lemma 5.4 to obtain a contradiction. Therefore, (5.3) is a finite sum, and sf is well-defined.

We also need to show that $\operatorname{supp}(sf)$ has DCC. Suppose to the contrary that

$$-\mathbf{a}^{(1)} + \mathbf{b}^{(1)}/q > -\mathbf{a}^{(2)} + \mathbf{b}^{(2)}/q > -\mathbf{a}^{(3)} + \mathbf{b}^{(3)}/q > \cdots.$$

is an infinite chain in $\operatorname{supp}(sf)$. If there are only finitely many distinct n-tuples $\mathbf{b}^{(k)}/q$, then we also obtain an infinite descending chain in the $-\mathbf{a}^{(k)}$, for $k \gg 0$, a contradiction since $\operatorname{supp}(f)$ has DCC. We may then assume that there are infinitely many $\mathbf{b}^{(k)}/q$ and all are distinct and then apply Lemma 5.4 again to obtain a contradiction. □

Lemma 5.4. *Let A and B be subsets of G^n, where $(G,+)$ is a linearly ordered abelian group. Suppose that A has DCC and that B has DCC in each coordinate. If $\{\mathbf{a}^{(k)}\}_k$ is a sequence of n-tuples in A and $\{\mathbf{b}^{(k)}\}_k$ is a sequence of infinitely many distinct n-tuples in B, then we cannot obtain an infinite chain*

$$\mathbf{a}^{(1)}+\mathbf{b}^{(1)} \geq \mathbf{a}^{(2)}+\mathbf{b}^{(2)} \geq \mathbf{a}^{(3)}+\mathbf{b}^{(3)} \geq \cdots . \tag{5.5}$$

Proof. Suppose we have an infinite chain as in (5.5). Because each $\mathbf{b}^{(k)}$ has only finitely many coordinates and each $\mathbf{b}^{(k)}$ is distinct, after taking subsequences, we may assume without loss of generality that, for each i, either $b_i^{(k)} = b_i^{(k')}$, or $b_i^{(k)} < b_i^{(k+1)}$, for all k, k'. (The latter assumption may be made when there are infinitely many distinct values because B has DCC in each coordinate.) These conditions imply that $\mathbf{b}^{(1)} < \mathbf{b}^{(2)} < \mathbf{b}^{(3)} < \cdots$. If subtract this chain of inequalities from (5.5), we obtain an infinite descending chain $\mathbf{a}^{(1)} > \mathbf{a}^{(2)} > \mathbf{a}^{(3)} > \cdots$, which contradicts the fact that A has DCC. □

Proposition 5.6. *Using the notation of Definition 5.1, N is an essential extension of K^∞. Therefore, $N \subseteq E_{R^\infty}(K^\infty)$.*

Proof. The second claim follows immediately from the first. For the first, let $f = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{x}^{-\mathbf{a}} \in N$. Since $\operatorname{supp}(f)$ has DCC, we can choose a minimal element $-\mathbf{a}^{(0)}$. Then $\mathbf{x}^{\mathbf{a}^{(0)}} \in S$, and

$$\mathbf{x}^{\mathbf{a}^{(0)}} f = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}^{(0)}-\mathbf{a}} = c_{\mathbf{a}^{(0)}} \in K^\infty \setminus \{0\}$$

as $\mathbf{a}_i^{(0)} > \mathbf{a}_i$, for some i, for all $\mathbf{a} \neq \mathbf{a}^{(0)}$ in $\operatorname{supp}(f)$. □

Remark 5.7. M. McDermott showed in [McD, Proposition 5.1.1] that N is the entire injective hull of K^∞ over R^∞ in dimension 1. McDermott's proof covers the case $R = K[x]$, but the case $R = K[[x]]$ follows routinely. Whether the result is true for $\dim R \geq 2$ is unknown.

Proposition 5.8. *With the notation of Definition 5.1, if $n \geq 2$, then the injective hull $E_{R^\infty}(K^\infty)$ contains an element not killed by any power of $m_R = (\mathbf{x})R$.*

Proof. Let $f = \sum_e x_1^{-1/p^e} x_2^{-e}$. For $e < e'$, $-1/p^e < -1/p^{e'}$ and $-e > -e'$ so that all elements in $\operatorname{supp}(f)$ are incomparable. Hence, all chains in $\operatorname{supp}(f)$ have only one link, and $f \in N$, which injects into $E_{R^\infty}(K^\infty)$ by the last proposition. Now, let $t > 0$. Then $x_2^t f = \sum_e x_1^{-1/p^e} x_2^{n-e}$, and if $e_0 \geq t$, then $t - e_0 \leq 0$. Therefore, $x_2^t f \neq 0$, and $m_R^t f \neq 0$, for any $t > 0$. □

5.2. **The General Case.** We will now show how we can extend Proposition 5.8 to include complete local domains and standard graded K-algebra domains in positive characteristic. Moreover, we will also extend the result to one concerning the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$.

An injection of R-modules $N \to M$ is called *pure* if $W \otimes N \to W \otimes M$ is an injection for all R-modules W. When M/N is finitely presented, the map is pure if and only if the map splits; see [HR, Corollary 5.2]. When S is an R-algebra and $R \to S$ is pure as a map of R-modules, one calls S *pure* over R.

Lemma 5.9. *Let $R = \varinjlim_\alpha R_\alpha$, and let $S = \varinjlim_\alpha S_\alpha$ such that each S_α is pure over R_α. Then S is pure over R.*

If A is a regular ring of positive characteristic and R is a reduced module-finite extension of A, then A is a direct summand of R as an A-module; see [Ho, Theorem 1]. Thus, $A^{1/q}$ is a direct summand of $R^{1/q}$, for all $q = p^e$, and so the last lemma implies that R^∞ is pure over A^∞.

Theorem 5.10. *Let (R, m, K) be a complete local domain (resp., a standard graded K-algebra domain) of positive characteristic and Krull dimension $n \geq 2$. Then there exists an element of $E := E_{R^\infty}(K^\infty)$ that is not killed by any power of m.*

Proof. By the Cohen structure theorem, R is a module-finite extension of a formal power series ring $A = K[[x_1, \ldots, x_n]]$ (resp., by Noether normalization, R is a module-finite extension of the graded polynomial ring $A = K[x_1, \ldots, x_n]$). Since A is regular, R is pure over A, and so the last lemma implies that R^∞ is pure over A^∞. If we let $E_0 := E_{A^\infty}(K^\infty)$, then

$$K^\infty \hookrightarrow E_0 \hookrightarrow M := R^\infty \otimes_{A^\infty} E_0.$$

Since K^∞ is an R^∞-module, we can find an R^∞-submodule M' of M maximal with respect to not intersecting K^∞. Hence, M/M' is an essential extension of K^∞ as an R^∞-module. We can then extend M/M' to a maximal essential extension E of K^∞ over R^∞. Since the inclusion $K^\infty \to E$ factors through E_0 and since E_0 is an essential extension of K^∞ over A^∞, E_0 injects into E as a map of A^∞-modules. Since E_0 contains an element not killed by any power of the maximal ideal (resp., the homogeneous maximal ideal) m_A of A by Proposition 5.8, so does E. Since m_A is primary to m, the same element of E not killed by a power of m_A is also not killed by a power of m. □

We can also take advantage of the faithful flatness of A^+ or $A^{+\mathrm{GR}}$ over a regular ring A (see [HH2, p. 77]) to prove the existence of elements not killed by a power of the maximal ideal in the injective hull of the residue field over R^+ or $R^{+\mathrm{GR}}$.

Theorem 5.11. *Let (R, m, K) be a complete local domain (resp., a standard graded K-algebra domain) of positive characteristic and Krull dimension $n \geq 2$. Then there exists an element of $E := E_{R^+}(\overline{K})$ (resp., $E := E_{R^{+\mathrm{GR}}}(\overline{K})$) that is not killed by any power of m, where $\overline{K}$ is the algebraic closure of K.*

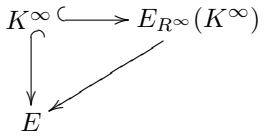
Proof. By the Cohen structure theorem, R is a module-finite extension of a formal power series ring $A = K[[x_1, \ldots, x_n]]$ (resp., by Noether normalization, R is a module-finite extension of the graded polynomial ring $A = K[x_1, \ldots, x_n]$). Thus, $A^+ \cong R^+$ (resp., $A^{+\mathrm{GR}} \cong R^{+\mathrm{GR}}$), and so we may assume that $R = K[[x_1, \ldots, x_n]]$ (resp., $R = K[x_1, \ldots, x_n]$). Let $B := R^+$ (resp., $B := R^{+\mathrm{GR}}$). Since $R^{1/q}$ is regular, for all q, and since B is a big Cohen-Macaulay $R^{1/q}$-algebra, B is faithfully flat over $R^{1/q}$. Therefore, B is flat over R^∞.

The inclusion of $K^\infty \subseteq E_{R^\infty}(K^\infty)$, together with the flatness of B over R^∞ gives the following diagram:

$$\begin{array}{ccc} K^\infty & \hookrightarrow & E_{R^\infty}(K^\infty) \\ \downarrow & & \downarrow \\ B \otimes_{R^\infty} K^\infty & \hookrightarrow & B \otimes_{R^\infty} E_{R^\infty}(K^\infty) \end{array}$$

As we have a surjection of $B \otimes_{R^\infty} K^\infty$ onto $\overline{K}$, the residue field of B, we have a map from $B \otimes_{R^\infty} K^\infty$ to E, the injective hull of $\overline{K}$ over B. Because E is injective,

this map lifts to a map from $B \otimes_{R^\infty} E_{R^\infty}(K^\infty)$. Hence, we obtain a commutative diagram of R^∞-module maps:



where the diagonal map is also injective since $E_{R^\infty}(K^\infty)$ is an essential extension of K^∞. Therefore, the element in $E_{R^\infty}(K^\infty)$ not killed by any power of m (as in Proposition 5.8) has a nonzero image in E that is not killed by any power of m. □

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E-mail address: gdietz@member.ams.org

Department of Mathematics, Gannon University, Erie, Pennsylvania 16541